tab.1  Soundness for Additional Rules

We say a rule is sound for a class of models if, whenever a branch in a tableau is satisfiable in a model from that class, the branch resulting from applying the rule is also satisfiable in a model from that class.

Proposition tab.1.  $\Box \psi$ and $\Diamond \psi$ are sound for reflexive models.

Proof.  1. The branch is expanded by applying $\Box \psi$ to $\sigma \Box \psi \in \Gamma$: This results in a new signed formula $\sigma \Box \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma) \vDash \Box \psi$. Since $R$ is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathcal{M}, f(\sigma) \vDash \psi$, i.e., $\mathcal{M}, f$ satisfies $\sigma \Box \psi$.

2. The branch is expanded by applying $\Diamond \psi$ to $\sigma \Diamond \psi \in \Gamma$: This results in a new signed formula $\sigma \Diamond \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma) \not\vDash \Diamond \psi$. Since $R$ is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathcal{M}, f(\sigma) \not\vDash \psi$, i.e., $\mathcal{M}, f$ satisfies $\sigma \Diamond \psi$. □

Proposition tab.2.  $D \Box$ and $D \Diamond$ are sound for serial models.

Proof.  1. The branch is expanded by applying $D \Box$ to $\sigma \Box \psi \in \Gamma$: This results in a new signed formula $\sigma \Box \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma) \vDash \Box \psi$. Since $R$ is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathcal{M}, w \vDash \psi$, and hence $\mathcal{M}, f(\sigma) \vDash \Diamond \psi$. So, $\mathcal{M}, f$ satisfies $\sigma \Box \psi$.

2. The branch is expanded by applying $D \Diamond$ to $\sigma \Diamond \psi \in \Gamma$: This results in a new signed formula $\sigma \Diamond \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma) \not\vDash \Diamond \psi$. Since $R$ is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathcal{M}, w \not\vDash \psi$, and hence $\mathcal{M}, f(\sigma) \not\vDash \Box \psi$. So, $\mathcal{M}, f$ satisfies $\sigma \Diamond \psi$. □

Proposition tab.3.  $B \Box$ and $B \Diamond$ are sound for symmetric models.

Proof.  1. The branch is expanded by applying $B \Box$ to $\sigma.n \Box \psi \in \Gamma$: This results in a new signed formula $\sigma \Box \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma.n) \vDash \Box \psi$. Since $f$ is an interpretation of prefixes on the branch into $\mathcal{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since $R$ is symmetric, $Rf(\sigma.n)f(\sigma)$. Hence, $\mathcal{M}, f$ satisfies $\sigma \Box \psi$.

2. The branch is expanded by applying $B \Diamond$ to $\sigma.n \Diamond \psi \in \Gamma$: This results in a new signed formula $\sigma \Diamond \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma.n) \not\vDash \Diamond \psi$. Since $f$ is an interpretation of prefixes on the branch into $\mathcal{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since $R$ is symmetric, $Rf(\sigma.n)f(\sigma)$. Hence, $\mathcal{M}, f$ satisfies $\sigma \Diamond \psi$. □
Proposition tab.4. 4□ and 4◊ are sound for transitive models.

Proof. 1. The branch is expanded by applying 4□ to σ □ ψ ∈ Γ: This results in a new signed formula σ.n □ ψ on the branch. Suppose M, f ⊩ Γ, in particular, M, f(σ) ⊩ □ ψ. Since f is an interpretation of prefixes on the branch into M, we know that Rf(σ)f(σ.n). Now let w be any world such that Rf(σ)n.w. Since R is transitive, Rf(σ).w. Since M, f(σ) ⊩ □ ψ, M, w ⊩ ψ. Hence, M, f(σ.n) ⊩ □ ψ, and M, f satisfies σ.n □ ψ.

2. The branch is expanded by applying 4◊ to σ F ◊ ψ ∈ Γ: This results in a new signed formula σ.n F ◊ ψ on the branch. Suppose M, f ⊩ Γ, in particular, M, f(σ) ⊮ ◊ ψ. Since f is an interpretation of prefixes on the branch into M, we know that Rf(σ)f(σ.n). Now let w be any world such that Rf(σ.n).w. Since R is transitive, Rf(σ).w. Since M, f(σ) ⊮ ◊ ψ, M, w ⊮ ψ. Hence, M, f(σ.n) ⊮ ◊ ψ, and M, f satisfies σ.n F ◊ ψ.

Proposition tab.5. 4r□ and 4r◊ are sound for euclidean models.

Proof. 1. The branch is expanded by applying 4r□ to σ □ ψ ∈ Γ: This results in a new signed formula σ.n □ ψ on the branch. Suppose M, f ⊩ Γ, in particular, M, f(σ) ⊩ □ ψ. Since f is an interpretation of prefixes on the branch into M, we know that Rf(σ)f(σ.n). Now let w be any world such that Rf(σ)n.w. Since R is euclidean, Rf(σ).w. Since M, f(σ) ⊩ □ ψ, M, w ⊩ ψ. Hence, M, f(σ.n) ⊩ □ ψ, and M, f satisfies σ.n □ ψ.

2. The branch is expanded by applying 4r◊ to σ F ◊ ψ ∈ Γ: This results in a new signed formula σ.n F ◊ ψ on the branch. Suppose M, f ⊩ Γ, in particular, M, f(σ) ⊮ ◊ ψ. Since f is an interpretation of prefixes on the branch into M, we know that Rf(σ)f(σ.n). Now let w be any world such that Rf(σ)n.w. Since R is euclidean, Rf(σ).w. Since M, f(σ) ⊮ ◊ ψ, M, w ⊮ ψ. Hence, M, f(σ)n ⊮ ◊ ψ, and M, f satisfies σ.n F ◊ ψ.

Corollary tab.6. The tableau systems given in ?? are sound for the respective classes of models.

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Bibliography