tab.1 Soundness for Additional Rules

We say a rule is sound for a class of models if, whenever a branch in a tableau is satisfiable in a model from that class, the branch resulting from applying the rule is also satisfiable in a model from that class.

Proposition tab.1. $\Box$ and $\Diamond$ are sound for reflexive models.

Proof. 1. The branch is expanded by applying $\Box$ to $\sigma \Box \psi \in \Gamma$: This results in a new signed formula $\sigma \Box \psi$ on the branch. Suppose $\mathcal{M}, f \models \Gamma$, in particular, $\mathcal{M}, f(\sigma) \models \Box \psi$. Since $R$ is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathcal{M}, f(\sigma) \models \psi$, i.e., $\mathcal{M}, f$ satisfies $\sigma \Box \psi$.

2. The branch is expanded by applying $\Diamond$ to $\sigma \Diamond \psi \in \Gamma$: This results in a new signed formula $\sigma \Diamond \psi$ on the branch. Suppose $\mathcal{M}, f \models \Gamma$, in particular, $\mathcal{M}, f(\sigma) \not\models \Diamond \psi$. Since $R$ is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathcal{M}, f(\sigma) \not\models \psi$, i.e., $\mathcal{M}, f$ satisfies $\sigma \Diamond \psi$.

Proposition tab.2. $\Box$ and $\Diamond$ are sound for serial models.

Proof. 1. The branch is expanded by applying $\Box$ to $\sigma \Box \psi \in \Gamma$: This results in a new signed formula $\sigma \Box \psi$ on the branch. Suppose $\mathcal{M}, f \models \Gamma$, in particular, $\mathcal{M}, f(\sigma) \models \Box \psi$. Since $R$ is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathcal{M}, w \models \psi$, and hence $\mathcal{M}, f(\sigma) \models \psi$. So, $\mathcal{M}, f$ satisfies $\sigma \Box \psi$.

2. The branch is expanded by applying $\Diamond$ to $\sigma \Diamond \psi \in \Gamma$: This results in a new signed formula $\sigma \Diamond \psi$ on the branch. Suppose $\mathcal{M}, f \models \Gamma$, in particular, $\mathcal{M}, f(\sigma) \not\models \Diamond \psi$. Since $R$ is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathcal{M}, w \not\models \psi$, and hence $\mathcal{M}, f(\sigma) \not\models \psi$. So, $\mathcal{M}, f$ satisfies $\sigma \Diamond \psi$.

Proposition tab.3. $\Box$ and $\Box$ are sound for symmetric models.

Proof. 1. The branch is expanded by applying $\Box$ to $\sigma.n \Box \psi \in \Gamma$: This results in a new signed formula $\sigma \Box \psi$ on the branch. Suppose $\mathcal{M}, f \models \Gamma$, in particular, $\mathcal{M}, f(\sigma,n) \models \Box \psi$. Since $f$ is an interpretation of prefixes on the branch into $\mathcal{M}$, we know that $Rf(\sigma)f(\sigma,n)$. Since $R$ is symmetric, $Rf(\sigma)n$ and $Rf(\sigma)f(\sigma)$. Hence, $\mathcal{M}, f$ satisfies $\sigma \Box \psi$.

2. The branch is expanded by applying $\Box$ to $\sigma.n \Box \psi \in \Gamma$: This results in a new signed formula $\sigma \Box \psi$ on the branch. Suppose $\mathcal{M}, f \models \Gamma$, in particular, $\mathcal{M}, f(\sigma,n) \not\models \Box \psi$. Since $f$ is an interpretation of prefixes on the branch into $\mathcal{M}$, we know that $Rf(\sigma)f(\sigma,n)$. Since $R$ is symmetric, $Rf(\sigma)n$ and $Rf(\sigma)f(\sigma)$. Hence, $\mathcal{M}, f$ satisfies $\sigma \Box \psi$.
Proposition tab.4. 4□ and 4♦ are sound for transitive models.

Proof. 1. The branch is expanded by applying 4□ to $\sigma \top \square \psi \in \Gamma$: This results in a new signed formula $\sigma.n \top \square \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma) \vDash \square \psi$. Since $f$ is an interpretation of prefixes on the branch into $\mathcal{M}$ and $\sigma.n$ must be used, we know that $RF(\sigma)f(\sigma.n)$. Now let $w$ be any world such that $RF(\sigma.n)w$. Since $R$ is transitive, $RF(\sigma)w$. Since $\mathcal{M}, f(\sigma) \vDash \square \psi$, $\mathcal{M}, w \vDash \psi$. Hence, $\mathcal{M}, f(\sigma.n) \vDash \square \psi$, and $\mathcal{M}, f$ satisfies $\sigma.n \top \square \psi$.

2. The branch is expanded by applying 4♦ to $\sigma \triangledown \diamond \psi \in \Gamma$: This results in a new signed formula $\sigma.n \top \diamond \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma) \nvdash \diamond \psi$. Since $f$ is an interpretation of prefixes on the branch into $\mathcal{M}$, we know that $RF(\sigma)f(\sigma.n)$. Now let $w$ be any world such that $RF(\sigma.n)w$. Since $R$ is transitive, $RF(\sigma)w$. Since $\mathcal{M}, f(\sigma) \nvdash \diamond \psi$, $\mathcal{M}, w \nvdash \psi$. Hence, $\mathcal{M}, f(\sigma.n) \nvdash \diamond \psi$, and $\mathcal{M}, f$ satisfies $\sigma.n \top \diamond \psi$.

Proposition tab.5. 4r□ and 4r♦ are sound for euclidean models.

Proof. 1. The branch is expanded by applying 4r□ to $\sigma \top \square \psi \in \Gamma$: This results in a new signed formula $\sigma.n \top \square \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma) \vDash \square \psi$. Since $f$ is an interpretation of prefixes on the branch into $\mathcal{M}$ and $\sigma.n$ must be used, we know that $RF(\sigma)f(\sigma.n)$. Now let $w$ be any world such that $RF(\sigma.n)w$. Since $R$ is euclidean, $RF(\sigma.n)w$ since $\mathcal{M}, f(\sigma) \vDash \square \psi$, $\mathcal{M}, w \vDash \psi$. Hence, $\mathcal{M}, f(\sigma.n) \vDash \square \psi$, and $\mathcal{M}, f$ satisfies $\sigma.n \top \square \psi$.

2. The branch is expanded by applying 4r♦ to $\sigma \triangledown \diamond \psi \in \Gamma$: This results in a new signed formula $\sigma.n \top \diamond \psi$ on the branch. Suppose $\mathcal{M}, f \vDash \Gamma$, in particular, $\mathcal{M}, f(\sigma) \nvdash \diamond \psi$. Since $f$ is an interpretation of prefixes on the branch into $\mathcal{M}$, we know that $RF(\sigma)f(\sigma.n)$. Now let $w$ be any world such that $RF(\sigma)n\wedge w$. Since $R$ is euclidean, $RF(\sigma)n\wedge w$. Since $\mathcal{M}, f(\sigma) \nvdash \diamond \psi$, $\mathcal{M}, w \nvdash \psi$. Hence, $\mathcal{M}, f(\sigma) \nvdash \diamond \psi$, and $\mathcal{M}, f$ satisfies $\sigma.n \top \diamond \psi$.

Corollary tab.6. The tableau systems given in ?? are sound for the respective classes of models.