

tab.1 Soundness for Additional Rules

mod:tab:msn:
sec

We say a rule is sound for a class of models if, whenever a branch in a **tableau** is satisfiable in a model from that class, the branch resulting from applying the rule is also satisfiable in a model from that class.

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prop:soundness-T

Proposition tab.1. $T\Box$ and $T\Diamond$ are sound for reflexive models.

Proof. 1. The branch is expanded by applying $T\Box$ to $\sigma T\Box\psi \in \Gamma$: This results in a new **signed formula** $\sigma T\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \psi$, i.e., \mathfrak{M}, f satisfies $\sigma T\psi$.

2. The branch is expanded by applying $T\Diamond$ to $\sigma F\Diamond\psi \in \Gamma$: This results in a new **signed formula** $\sigma F\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \not\Vdash \psi$, i.e., \mathfrak{M}, f satisfies $\sigma F\psi$. \square

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prop:soundness-D

Proposition tab.2. $D\Box$ and $D\Diamond$ are sound for serial models.

Proof. 1. The branch is expanded by applying $D\Box$ to $\sigma T\Box\psi \in \Gamma$: This results in a new **signed formula** $\sigma T\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \Vdash \psi$, and hence $\mathfrak{M}, f(\sigma) \Vdash \Diamond\psi$. So, \mathfrak{M}, f satisfies $\sigma T\Diamond\psi$.

2. The branch is expanded by applying $D\Diamond$ to $\sigma F\Diamond\psi \in \Gamma$: This results in a new **signed formula** $\sigma F\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \not\Vdash \psi$, and hence $\mathfrak{M}, f(\sigma) \not\Vdash \Box\psi$. So, \mathfrak{M}, f satisfies $\sigma F\Box\psi$. \square

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prop:soundness-B

Proposition tab.3. $B\Box$ and $B\Diamond$ are sound for symmetric models.

Proof. 1. The branch is expanded by applying $B\Box$ to $\sigma.n T\Box\psi \in \Gamma$: This results in a new **signed formula** $\sigma T\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into \mathfrak{M} , we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, $\mathfrak{M}, f(\sigma) \Vdash \psi$. Hence, \mathfrak{M}, f satisfies $\sigma T\psi$.

2. The branch is expanded by applying $B\Diamond$ to $\sigma.n F\Diamond\psi \in \Gamma$: This results in a new **signed formula** $\sigma F\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into \mathfrak{M} , we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$, $\mathfrak{M}, f(\sigma) \not\Vdash \psi$. Hence, \mathfrak{M}, f satisfies $\sigma F\psi$. \square

mod:tab:msn:
prop:soundness-4

Proposition tab.4. $4\Box$ and $4\Diamond$ are sound for transitive models.

- Proof.* 1. The branch is expanded by applying $4\Box$ to $\sigma\mathbb{T}\Box\psi \in \Gamma$: This results in a new **signed formula** $\sigma.n\mathbb{T}\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into \mathfrak{M} and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, and \mathfrak{M}, f satisfies $\sigma.n\mathbb{T}\Box\psi$.
2. The branch is expanded by applying $4\Diamond$ to $\sigma\mathbb{F}\Diamond\psi \in \Gamma$: This results in a new **signed formula** $\sigma.n\mathbb{F}\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into \mathfrak{M} and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$, $\mathfrak{M}, w \not\Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$, and \mathfrak{M}, f satisfies $\sigma.n\mathbb{F}\Diamond\psi$. \square

Proposition tab.5. $4r\Box$ and $4r\Diamond$ are sound for euclidean models.

*mod:tab:msn:
prop:soundness-4r*

- Proof.* 1. The branch is expanded by applying $4r\Box$ to $\sigma.n\mathbb{T}\Box\psi \in \Gamma$: This results in a new **signed formula** $\sigma\mathbb{T}\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into \mathfrak{M} , we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, and \mathfrak{M}, f satisfies $\sigma\mathbb{T}\Box\psi$.
2. The branch is expanded by applying $4r\Diamond$ to $\sigma.n\mathbb{F}\Diamond\psi \in \Gamma$: This results in a new **signed formula** $\sigma\mathbb{T}\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into \mathfrak{M} , we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$, $\mathfrak{M}, w \not\Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$, and \mathfrak{M}, f satisfies $\sigma\mathbb{F}\Diamond\psi$. \square

Corollary tab.6. The *tableau* systems given in ?? are sound for the respective classes of models.

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cor:soundness-logics*

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Bibliography