The proof of the completeness theorem doesn’t just show that if $\vDash \varphi$ then $\vdash \varphi$, it also gives us a method for constructing countermodels to $\varphi$ if $\nvDash A$. In the case of $K$, this method constitutes a decision procedure. For suppose $\nvDash \varphi$. Then the proof of ?? gives a method for constructing a complete tableau. The method in fact always terminates. The propositional rules for $K$ only add prefixed formulas of lower complexity, i.e., each propositional rule need only be applied once on a branch for any signed formula $\sigma S \varphi$. New prefixes are only generated by the $\Box F$ and $\Diamond T$ rules, and also only have to be applied once (and produce a single new prefix). $\Box T$ and $\Diamond F$ have to be applied potentially multiple times, but only once per prefix, and only finitely many new prefixes are generated. So the construction either results in a closed branch or a complete branch after finitely many stages.

Once a tableau with an open complete branch is constructed, the proof of ?? gives us an explicit model that satisfies the original set of prefixed formulas. So not only is it the case that if $\Gamma \vDash \varphi$, then a closed tableau exists and $\Gamma \vdash \varphi$, if we look for the closed tableau in the right way and end up with a “complete” tableau, we’ll not only know that $\Gamma \nvDash \varphi$ but actually be able to construct a countermodel.

**Example tab.1.** We know that $\nvDash \Box (p \lor q) \rightarrow (\Box p \lor \Box q)$. The construction of a tableau begins with:

<table>
<thead>
<tr>
<th>Step</th>
<th>Line</th>
<th>Formula</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>F</td>
<td>$\Box (p \lor q) \rightarrow (\Box p \lor \Box q)$ ✓</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>T</td>
<td>$\Box (p \lor q)$</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>F</td>
<td>$\Box p \lor \Box q$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>4</td>
<td>F</td>
<td>$\Box p$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>5</td>
<td>F</td>
<td>$\Box q$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>6</td>
<td>1.1 F</td>
<td>$p$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>7</td>
<td>1.2 F</td>
<td>$q$ ✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

The tableau is of course not finished yet. In the next step, we consider the only line without a checkmark: the prefixed formula $1 \ T \Box (p \lor q)$ on line 2. The construction of the closed tableau says to apply the $\Box T$ rule for every prefix used on the branch, i.e., for both 1.1 and 1.2:

<table>
<thead>
<tr>
<th>Step</th>
<th>Line</th>
<th>Formula</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>F</td>
<td>$\Box (p \lor q) \rightarrow (\Box p \lor \Box q)$ ✓</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>T</td>
<td>$\Box (p \lor q)$</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>F</td>
<td>$\Box p \lor \Box q$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>4</td>
<td>F</td>
<td>$\Box p$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>5</td>
<td>F</td>
<td>$\Box q$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>6</td>
<td>1.1 F</td>
<td>$p$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>7</td>
<td>1.2 F</td>
<td>$q$ ✓</td>
<td>✓</td>
</tr>
<tr>
<td>8</td>
<td>1.1 T</td>
<td>$p \lor q$</td>
<td>✓</td>
</tr>
<tr>
<td>9</td>
<td>1.2 T</td>
<td>$p \lor q$</td>
<td>✓</td>
</tr>
</tbody>
</table>
Now lines 2, 8, and 9, don’t have checkmarks. But no new prefix has been added, so we apply $\lor T$ to lines 8 and 9, on all resulting branches (as long as they don’t close):

There is one remaining open branch, and it is complete. From it we define the model with worlds $W = \{1, 1.1, 1.2\}$ (the only prefixes appearing on the open branch), the accessibility relation $R = \{(1, 1.1), (1, 1.2)\}$, and the assignment $V(p) = \{1.2\}$ (because line 11 contains $1.2 T p$) and $V(q) = \{1.1\}$ (because line 10 contains $1.1 T q$). The model is pictured in Figure 1, and you can verify that it is a countermodel to $\Box(p \lor q) \rightarrow (\Box p \lor \Box q)$.

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Bibliography