

tab.1 Countermodels from Tableaux

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sec

The proof of the completeness theorem doesn't just show that if $\models \varphi$ then $\vdash \varphi$, it also gives us a method for constructing countermodels to φ if $\not\models A$. In the case of **K**, this method constitutes a *decision procedure*. For suppose $\not\models \varphi$. Then the proof of ?? gives a method for constructing a complete **tableau**. The method in fact always terminates. The propositional rules for **K** only add prefixed **formulas** of lower complexity, i.e., each propositional rule need only be applied once on a branch for any signed formula $\sigma S\varphi$. New prefixes are only generated by the $\Box\mathbb{F}$ and $\Diamond\mathbb{T}$ rules, and also only have to be applied once (and produce a single new prefix). $\Box\mathbb{T}$ and $\Diamond\mathbb{F}$ have to be applied potentially multiple times, but only once per prefix, and only finitely many new prefixes are generated. So the construction either results in a closed branch or a complete branch after finitely many stages.

Once a tableau with an open complete branch is constructed, the proof of ?? gives us an explicit model that satisfies the original set of prefixed **formulas**. So not only is it the case that if $\Gamma \models \varphi$, then a closed **tableau** exists and $\Gamma \vdash \varphi$, if we look for the closed **tableau** in the right way and end up with a “complete” **tableau**, we'll not only know that $\Gamma \not\models \varphi$ but actually be able to construct a countermodel.

Example tab.1. We know that $\not\models \Box(p \vee q) \rightarrow (\Box p \vee \Box q)$. The construction of a tableau begins with:

1.	1 \mathbb{F} $\Box(p \vee q) \rightarrow (\Box p \vee \Box q) \checkmark$	Assumption
2.	1 \mathbb{T} $\Box(p \vee q)$	$\rightarrow\mathbb{F}$ 1
3.	1 \mathbb{F} $\Box p \vee \Box q \checkmark$	$\rightarrow\mathbb{F}$ 1
4.	1 \mathbb{F} $\Box p \checkmark$	$\vee\mathbb{F}$ 3
5.	1 \mathbb{F} $\Box q \checkmark$	$\vee\mathbb{F}$ 3
6.	1.1 \mathbb{F} $p \checkmark$	$\Box\mathbb{F}$ 4
7.	1.2 \mathbb{F} $q \checkmark$	$\Box\mathbb{F}$ 5

The **tableau** is of course not finished yet. In the next step, we consider the only line without a checkmark: the prefixed **formula** 1 $\mathbb{T}\Box(p \vee q)$ on line 2. The construction of the closed tableau says to apply the $\Box\mathbb{T}$ rule for every prefix used on the branch, i.e., for both 1.1 and 1.2:

1.	1 \mathbb{F} $\Box(p \vee q) \rightarrow (\Box p \vee \Box q) \checkmark$	Assumption
2.	1 \mathbb{T} $\Box(p \vee q)$	$\rightarrow\mathbb{F}$ 1
3.	1 \mathbb{F} $\Box p \vee \Box q \checkmark$	$\rightarrow\mathbb{F}$ 1
4.	1 \mathbb{F} $\Box p \checkmark$	$\vee\mathbb{F}$ 3
5.	1 \mathbb{F} $\Box q \checkmark$	$\vee\mathbb{F}$ 3
6.	1.1 \mathbb{F} $p \checkmark$	$\Box\mathbb{F}$ 4
7.	1.2 \mathbb{F} $q \checkmark$	$\Box\mathbb{F}$ 5
8.	1.1 \mathbb{T} $p \vee q$	$\Box\mathbb{T}$ 2
9.	1.2 \mathbb{T} $p \vee q$	$\Box\mathbb{T}$ 2

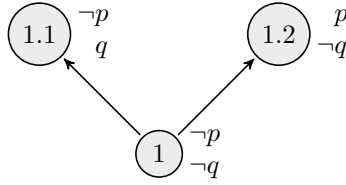


Figure 1: A countermodel to $\Box(p \vee q) \rightarrow (\Box p \vee \Box q)$.

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fig:counter-Box

Now lines 2, 8, and 9, don't have checkmarks. But no new prefix has been added, so we apply $\forall\mathbb{T}$ to lines 8 and 9, on all resulting branches (as long as they don't close):

1.	1F $\Box(p \vee q) \rightarrow (\Box p \vee \Box q) \checkmark$	Assumption
2.	1T $\Box(p \vee q)$	$\rightarrow\mathbb{F} 1$
3.	1F $\Box p \vee \Box q \checkmark$	$\rightarrow\mathbb{F} 1$
4.	1F $\Box p \checkmark$	$\forall\mathbb{F} 3$
5.	1F $\Box q \checkmark$	$\forall\mathbb{F} 3$
6.	1.1F $p \checkmark$	$\Box\mathbb{F} 4$
7.	1.2F $q \checkmark$	$\Box\mathbb{F} 5$
8.	1.1T $p \vee q \checkmark$	$\Box\mathbb{T} 2$
9.	1.2T $p \vee q \checkmark$	$\Box\mathbb{T} 2$
<div style="display: flex; justify-content: space-around; width: 100%;"> <div style="text-align: center;"> \swarrow \otimes </div> <div style="text-align: center;"> \searrow \otimes </div> </div>		
10.	1.1T $p \checkmark$	$\forall\mathbb{T} 8$
11.	1.2T $p \checkmark$	$\forall\mathbb{T} 9$

There is one remaining open branch, and it is complete. From it we define the model with worlds $W = \{1, 1.1, 1.2\}$ (the only prefixes appearing on the open branch), the accessibility relation $R = \{(1, 1.1), (1, 1.2)\}$, and the assignment $V(p) = \{1.2\}$ (because line 11 contains $1.2\mathbb{T}p$) and $V(q) = \{1.1\}$ (because line 10 contains $1.1\mathbb{T}q$). The model is pictured in Figure 1, and you can verify that it is a countermodel to $\Box(p \vee q) \rightarrow (\Box p \vee \Box q)$.

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Bibliography