The proof of the completeness theorem doesn’t just show that if $\models \varphi$ then $\vdash \varphi$, it also gives us a method for constructing countermodels to $\varphi$ if $\not\models A$. In the case of $K$, this method constitutes a decision procedure. For suppose $\not\models \varphi$. Then the proof of $??$ gives a method for constructing a complete tableau. The method in fact always terminates. The propositional rules for $K$ only add prefixed formulas of lower complexity, i.e., each propositional rule need only be applied once on a branch for any signed formula $\sigma S \varphi$. New prefixes are only generated by the $\Box F$ and $\Diamond T$ rules, and also only have to be applied once (and produce a single new prefix). $\Box T$ and $\Diamond F$ have to be applied potentially multiple times, but only once per prefix, and only finitely many new prefixes are generated. So the construction either results in a closed branch or a complete branch after finitely many stages.

Once a tableau with an open complete branch is constructed, the proof of $??$ gives us an explicit model that satisfies the original set of prefixed formulas. So not only is it the case that if $\Gamma \models \varphi$, then a closed tableau exists and $\Gamma \vdash \varphi$, if we look for the closed tableau in the right way and end up with a “complete” tableau, we’ll not only know that $\Gamma \not\models \varphi$ but actually be able to construct a countermodel.

**Example tab.1.** We know that $\not\models \Box (p \lor q) \rightarrow (\Box p \lor \Box q)$. The construction of a tableau begins with:

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|   | 1 | $F$ | $\Box (p \lor q) \rightarrow (\Box p \lor \Box q) \checkmark$ | Assumption |
| 2 | 1 | $T$ | $\Box (p \lor q)$ | $\rightarrow F$ 1 |
| 3 | 1 | $F$ | $\Box p \lor q \checkmark$ | $\rightarrow F$ 1 |
| 4 | 1 | $F$ | $\Box p \checkmark$ | $\lor F$ 3 |
| 5 | 1 | $F$ | $\Box q \checkmark$ | $\lor F$ 3 |
| 6 | 1.1 | $F$ | $p \checkmark$ | $\Box F$ 4 |
| 7 | 1.2 | $F$ | $q \checkmark$ | $\Box F$ 5 |

The tableau is of course not finished yet. In the next step, we consider the only line without a checkmark: the prefixed formula $1T \Box (p \lor q)$ on line 2. The construction of the closed tableau says to apply the $\Box T$ rule for every prefix used on the branch, i.e., for both 1.1 and 1.2:

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|   | 1 | $F$ | $\Box (p \lor q) \rightarrow (\Box p \lor \Box q) \checkmark$ | Assumption |
| 2 | 1 | $T$ | $\Box (p \lor q)$ | $\rightarrow F$ 1 |
| 3 | 1 | $F$ | $\Box p \lor \Box q \checkmark$ | $\rightarrow F$ 1 |
| 4 | 1 | $F$ | $\Box p \checkmark$ | $\lor F$ 3 |
| 5 | 1 | $F$ | $\Box q \checkmark$ | $\lor F$ 3 |
| 6 | 1.1 | $F$ | $p \checkmark$ | $\Box F$ 4 |
| 7 | 1.2 | $F$ | $q \checkmark$ | $\Box F$ 5 |
| 8 | 1.1 | $T$ | $p \lor q$ | $\Box T$ 2 |
| 9 | 1.2 | $T$ | $p \lor q$ | $\Box T$ 2 |
 Figure 1: A countermodel to $\Box (p \lor q) \rightarrow (\Box p \lor \Box q)$. 

Now lines 2, 8, and 9, don’t have checkmarks. But no new prefix has been added, so we apply $\lor \top$ to lines 8 and 9, on all resulting branches (as long as they don’t close):

1. $\top \; F \; \Box (p \lor q) \rightarrow (\Box p \lor \Box q) \; \checkmark$ Assumption
2. $\top \; T \; \Box (p \lor q) \rightarrow \Box 1$
3. $\top \; F \; \Box p \lor \Box q \; \checkmark \rightarrow \Box 1$
4. $\top \; F \; \Box p \; \checkmark \lor \Box 3$
5. $\top \; F \; \Box q \; \checkmark \lor \Box 3$
6. $\top \; F \; \Box p \lor \Box q \; \checkmark \lor \Box 4$
7. $\top \; T \; p \lor q \; \checkmark \lor \Box 5$
8. $\top \; T \; p \lor q \; \checkmark \lor \Box 2$
9. $\top \; T \; p \lor q \; \checkmark \lor \Box 2$
10. $\top \; T \; p \lor q \; \checkmark \lor \Box 8$
11. $\top \; T \; p \lor q \; \checkmark \lor \Box 9$

There is one remaining open branch, and it is complete. From it we define the model with worlds $W = \{1, 1.1, 1.2\}$ (the only prefixes appearing on the open branch), the accessibility relation $R = \{(1, 1.1), (1, 1.2)\}$, and the assignment $V(p) = \{1.2\}$ (because line 10 contains $1.2 T p$) and $V(q) = \{1.1\}$ (because line 10 contains $1.1 T q$). The model is pictured in Figure 1, and you can verify that it is a countermodel to $\Box (p \lor q) \rightarrow (\Box p \lor \Box q)$.

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Bibliography