tab.1 Countermodels from Tableaux

mod:tab:cou:sec The proof of the completeness theorem doesn't just show that if $\vDash \varphi$ then $\vdash \varphi$, it also gives us a method for constructing countermodels to φ if $\nvDash A$. In the case of **K**, this method constitutes a *decision procedure*. For suppose $\nvDash \varphi$. Then the proof of **??** gives a method for constructing a complete tableau. The method in fact always terminates. The propositional rules for **K** only add prefixed formulas of lower complexity, i.e., each propositional rule need only be applied once on a branch for any signed formula $\sigma\, S\, \varphi$. New prefixes are only generated by the $\Box\mathbb{F}$ and $\Diamond\mathbb{T}$ rules, and also only have to be applied once (and produce a single new prefix). $\Box\mathbb{T}$ and $\Diamond\mathbb{F}$ have to be applied potentially multiple times, but only once per prefix, and only finitely many new prefixes are generated. So the construction either results in a closed branch or a complete branch after finitely many stages. explanation

Once a tableau with an open complete branch is constructed, the proof of **??** gives us an explict model that satisfies the original set of prefixed formulas. So not only is it the case that if $\Gamma \vDash \varphi$, then a closed tableau exists and $\Gamma \vdash \varphi$, if we look for the closed tableau in the right way and end up with a "complete" tableau, we'll not only know that $\Gamma \nvDash \varphi$ but actually be able to construct a countermodel.

Example tab.1. We know that $\nvdash \Box(p \lor q) \to (\Box p \lor \Box q)$. The construction of a tableau begins with:

1.	$1\,\mathbb{F}\ \ \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓	Assumption
2.	$1\,\mathbb{T}\ \ \Box(p \lor q)$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Box p \lor \Box q$ ✓	$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\ \ \Box p$ ✓	$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \ \Box q$ ✓	$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\ \ p$ ✓	$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\ \ q$ ✓	$\Box\mathbb{F}\,5$

The tableau is of course not finished yet. In the next step, we consider the only line without a checkmark: the prefixed formula $1\,\mathbb{T}\,\Box(p \lor q)$ on line 2. The construction of the closed tableau says to apply the $\Box\mathbb{T}$ rule for every prefix used on the branch, i.e., for both 1.1 and 1.2:

1.	$1\,\mathbb{F}\ \ \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓	Assumption
2.	$1\,\mathbb{T}\ \ \Box(p \lor q)$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Box p \lor \Box q$ ✓	$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\ \ \Box p$ ✓	$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \ \Box q$ ✓	$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\ \ p$ ✓	$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\ \ q$ ✓	$\Box\mathbb{F}\,5$
8.	$1.1\,\mathbb{T}\ \ p \lor q$	$\Box\mathbb{T}\,2$
9.	$1.2\,\mathbb{T}\ \ p \lor q$	$\Box\mathbb{T}\,2$

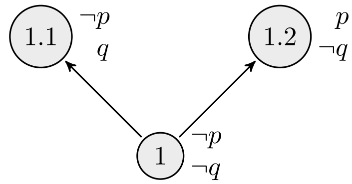

Figure 1: A countermodel to $\Box(p \lor q) \to (\Box p \lor \Box q)$.

mod:tab:cou: fig:counter-Box

Now lines 2, 8, and 9, don't have checkmarks. But no new prefix has been added, so we apply $\lor\mathbb{T}$ to lines 8 and 9, on all resulting branches (as long as they don't close):

1.	$1\,\mathbb{F}\;\; \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓			Assumption
2.	$1\,\mathbb{T}\;\; \Box(p \lor q)$			$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\;\; \Box p \lor \Box q$ ✓			$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\;\; \Box p$ ✓			$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\;\; \Box q$ ✓			$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\;\; p$ ✓			$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\;\; q$ ✓			$\Box\mathbb{F}\,5$
8.	$1.1\,\mathbb{T}\;\; p \lor q$ ✓			$\Box\mathbb{T}\,2$
9.	$1.2\,\mathbb{T}\;\; p \lor q$ ✓			$\Box\mathbb{T}\,2$
10.	$1.1\,\mathbb{T}\;\; p$ ✓ $\otimes$	$1.1\,\mathbb{T}\;\; q$ ✓		$\lor\mathbb{T}\,8$
11.		$1.2\,\mathbb{T}\;\; p$ ✓	$1.2\,\mathbb{T}\;\; q$ ✓ $\otimes$	$\lor\mathbb{T}\,9$

There is one remaining open branch, and it is complete. From it we define the model with worlds $W = \{1, 1.1, 1.2\}$ (the only prefixes appearing on the open branch), the accessibility relation $R = \{\langle 1, 1.1\rangle, \langle 1, 1.2\rangle\}$, and the assignment $V(p) = \{1.2\}$ (because line 11 contains $1.2\,\mathbb{T}\,p$) and $V(q) = \{1.1\}$ (because line 10 contains $1.1\,\mathbb{T}\,q$). The model is pictured in Figure 1, and you can verify that it is a countermodel to $\Box(p \lor q) \to (\Box p \lor \Box q)$.

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Bibliography