tab.1 Completeness for K

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To show that the method of tableaux is complete, we have to show that whenever there is no closed tableau to show $\Gamma \vdash \varphi$, then $\Gamma \nvDash \varphi$, i.e., there is a countermodel. But "there is no closed tableau" means that every way we could try to construct one has to fail to close. The trick is to see that if every such way fails to close, then a specific, *systematic and exhaustive* way also fails to close. And this systematic and exhaustive way would close if a closed tableau exists. The single tableau will contain, among its open branches, all the information required to define a countermodel. The countermodel given by an open branch in this tableau will contain the all the prefixes used on that branch as the worlds, and a propositional variable p is true at σ iff $\sigma\, \mathbb{T}\, p$ occurs on the branch. explanation

Definition tab.1. A branch in a tableau is called complete if, whenever it contains a prefixed formula $\sigma\, S\, \varphi$ to which a rule can be applied, it also contains

1. the prefixed formulas that are the corresponding conclusions of the rule, in the case of propositional stacking rules;
2. one of the corresponding conclusion formulas in the case of propositional branching rules;
3. at least one possible conclusion in the case of modal rules that require a new prefix;
4. the corresponding conclusion for every prefix occurring on the branch in the case of modal rules that require a used prefix.

For instance, a complete branch contains $\sigma\, \mathbb{T}\, \psi$ and $\sigma\, \mathbb{T}\, \chi$ whenever it contains $\mathbb{T}\, \psi \wedge \chi$. If it contains $\sigma\, \mathbb{T}\, \psi \vee \chi$ it contains at least one of $\sigma\, \mathbb{F}\, \psi$ and $\sigma\, \mathbb{T}\, \chi$. If it contains $\sigma\, \mathbb{F}\, \Box$ it also contains $\sigma.n\, \mathbb{F}\, \Box$ for at least one n. And whenever it contains $\sigma\, \mathbb{T}\, \Box$ it also contains $\sigma.n\, \mathbb{T}\, \Box$ for every n such that $\sigma.n$ is used on the branch. explanation

mod:tab:cpl: prop:complete-tableau

Proposition tab.2. *Every finite Γ has a tableau in which every branch is complete.*

Proof. Consider an open branch in a tableau for Γ. There are finitely many prefixed formulas in the branch to which a rule could be applied. In some fixed order (say, top to bottom), for each of these prefixed formulas for which the conditions (1)–(4) do not already hold, apply the rules that can be applied to it to extend the branch. In some cases this will result in branching; apply the rule at the tip of each resulting branch for all remaining prefixed formulas. Since the number of prefixed formulas is finite, and the number of used prefixes on the branch is finite, this procedure eventually results in (possibly many) branches extending the original branch. Apply the procedure to each, and repeat. But by construction, every branch is closed. □

Theorem tab.3 (Completeness)**.** *If Γ has no closed tableau, Γ is satisfiable.* *mod:tab:cpl: thm:tableau-completeness*

Proof. By the proposition, Γ has a tableau in which every branch is complete. Since it has no closed tableau, it thas has a tableau in which at least one branch is open and complete. Let Δ be the set of prefixed formulas on the branch, and $P(\Delta)$ the set of prefixes occurring in it.

We define a model $\mathfrak{M}(\Delta) = \langle P(\Delta), R, V\rangle$ where the worlds are the prefixes occurring in Δ, the accessibility relation is given by:

$$R\sigma\sigma' \quad \text{iff} \quad \sigma' = \sigma.n \quad \text{for some } n$$

and

$$V(p) = \{\sigma : \sigma\,\mathbb{T}\,p \in \Delta\}.$$

We show by induction on φ that if $\sigma\,\mathbb{T}\,\varphi \in \Delta$ then $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$, and if $\sigma\,\mathbb{F}\,\varphi \in \Delta$ then $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

1. $\varphi \equiv p$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$ then $\sigma \in V(p)$ (by definition of V) and so $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$ then $\sigma\,\mathbb{T}\,\varphi \notin \Delta$, since the branch would otherwise be closed. So $\sigma \notin V(p)$ and thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

2. $\varphi \equiv \neg\psi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then $\sigma\,\mathbb{F}\,\psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ and thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then $\sigma\,\mathbb{T}\,\psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

3. $\varphi \equiv \psi \wedge \varphi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then both $\sigma\,\mathbb{T}\,\psi \in \Delta$ and $\sigma\,\mathbb{T}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then either $\sigma\,\mathbb{F}\,\psi \in \Delta$ or $\sigma\,\mathbb{F}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

4. $\varphi \equiv \psi \vee \varphi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then either $\sigma\,\mathbb{T}\,\psi \in \Delta$ or $\sigma\,\mathbb{T}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then both $\sigma\,\mathbb{F}\,\psi \in \Delta$ and $\sigma\,\mathbb{F}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, both $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ and $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

5. $\varphi \equiv \psi \to \varphi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then either $\sigma\,\mathbb{F}\,\psi \in \Delta$ or $\sigma\,\mathbb{T}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then both $\sigma\,\mathbb{T}\,\psi \in \Delta$ and $\sigma\,\mathbb{F}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, both $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

6. $\varphi \equiv \Box\psi$: If $\sigma\, \mathbb{T}\, \varphi \in \Delta$, then, since the branch is complete, $\sigma.n\, \mathbb{T}\, \psi \in \Delta$ for every $\sigma.n$ used on the branch, i.e., for every $\sigma' \in P(\Delta)$ such that $R\sigma\sigma'$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma' \Vdash \psi$ for every σ' such that $R\sigma\sigma'$. Therefore, $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\, \mathbb{F}\, \varphi \in \Delta$, then for some $\sigma.n$, $\sigma.n\, \mathbb{F}\, \psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma.n \nVdash \psi$. Since $R\sigma(\sigma.n)$, there is a σ' such that $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

7. $\varphi \equiv \Diamond\psi$: If $\sigma\, \mathbb{T}\, \varphi \in \Delta$, then for some $\sigma.n$, $\sigma.n\, \mathbb{T}\, \psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$. Since $R\sigma(\sigma.n)$, there is a σ' such that $\mathfrak{M}(\Delta), \sigma' \Vdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\, \mathbb{F}\, \varphi \in \Delta$, then, since the branch is complete, $\sigma.n\, \mathbb{F}\, \psi \in \Delta$ for every $\sigma.n$ used on the branch, i.e., for every $\sigma' \in P(\Delta)$ such that $R\sigma\sigma'$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$ for every σ' such that $R\sigma\sigma'$. Therefore, $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

Since $\Gamma \subseteq \Delta$, $\mathfrak{M}(\Delta) \Vdash \Gamma$. □

Problem tab.1. Complete the proof of Theorem tab.3.

mod:tab:cpl: cor:entailment-completeness **Corollary tab.4.** *If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.*

mod:tab:cpl: cor:weak-completeness **Corollary tab.5.** *If φ is true in all models, then $\vdash \varphi$.*

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Bibliography