syn.1 Truth in a Model

nml:syn:tru:sec Sometimes we are interested in which formulas are true at every world in a given model. Let's introduce a notation for this.

Definition syn.1. A formula φ is *true in a model* $M = \langle W, R, V \rangle$, written $\mathfrak{M} \Vdash \varphi$, if and only if $\mathfrak{M}, w \Vdash \varphi$ for every $w \in W$.

nml:syn:tru:prop:truthfacts **Proposition syn.2.**

1. *If $\mathfrak{M} \Vdash \varphi$ then $\mathfrak{M} \nVdash \neg\varphi$, but* not *vice-versa.*

2. *If $\mathfrak{M} \Vdash \varphi \to \psi$ then $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but* not *vice-versa.*

Proof.

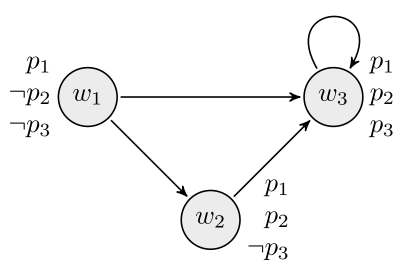
1. If $\mathfrak{M} \Vdash \varphi$ then φ is true at all worlds in W, and since $W \neq \emptyset$, it can't be that $\mathfrak{M} \Vdash \neg\varphi$, or else φ would have to be both true and false at some world.

   On the other hand, if $\mathfrak{M} \nVdash \neg\varphi$ then φ is true at some world $w \in W$. It does not follow that $\mathfrak{M}, w \Vdash \varphi$ for *every* $w \in W$. For instance, in the model of **??**, $\mathfrak{M} \nVdash \neg p$, and also $\mathfrak{M} \nVdash p$.

2. Assume $\mathfrak{M} \Vdash \varphi \to \psi$ and $\mathfrak{M} \Vdash \varphi$; to show $\mathfrak{M} \Vdash \psi$ let $w \in W$ be an arbitrary world. Then $\mathfrak{M}, w \Vdash \varphi \to \psi$ and $\mathfrak{M}, w \Vdash \varphi$, so $\mathfrak{M}, w \Vdash \psi$, and since w was arbitrary, $\mathfrak{M} \Vdash \psi$.

   To show that the converse fails, we need to find a model $\mathfrak{M}$ such that $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but $\mathfrak{M} \nVdash \varphi \to \psi$. Consider again the model of **??**: $\mathfrak{M} \nVdash p$ and hence (vacuously) $\mathfrak{M} \Vdash p$ only if $\mathfrak{M} \Vdash q$. However, $\mathfrak{M} \nVdash p \to q$, as p is true but q false at w_1. □

Problem syn.1. Consider the following model $\mathfrak{M}$ for the language comprising p_1, p_2, p_3 as the only propositional variables:



Are the following formulas and schemas true in the model $\mathfrak{M}$, i.e., true at every world in $\mathfrak{M}$? Explain.

1. $p \to \Diamond p$ (for p atomic);

2. $\varphi \to \Diamond\varphi$ (for φ arbitrary);

3. $\Box p \to p$ (for p atomic);
4. $\neg p \to \Diamond\Box p$ (for p atomic);
5. $\Diamond\Box\varphi$ (for φ arbitrary);
6. $\Box\Diamond p$ (for p atomic).

Photo Credits

Bibliography