

syn.1 Truth in a Model

mod:syn:tru:
sec

Sometimes we are interested which **formulas** are true at every world in a given model. Let's introduce a notation for this.

Definition syn.1. A **formula** φ is *true in a model* $M = \langle W, R, V \rangle$, written $\mathfrak{M} \Vdash \varphi$, if and only if $\mathfrak{M}, w \Vdash \varphi$ for every $w \in W$.

mod:syn:tru:
prop:truthfacts

Proposition syn.2.

1. If $\mathfrak{M} \Vdash \varphi$ then $\mathfrak{M} \not\Vdash \neg\varphi$, but not vice-versa.
2. If $\mathfrak{M} \Vdash \varphi \rightarrow \psi$ then $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but not vice-versa.

Proof. 1. If $\mathfrak{M} \Vdash \varphi$ then φ is true at all worlds in W , and since $W \neq \emptyset$, it can't be that $\mathfrak{M} \Vdash \neg\varphi$, or else φ would have to be both true and false at some world.

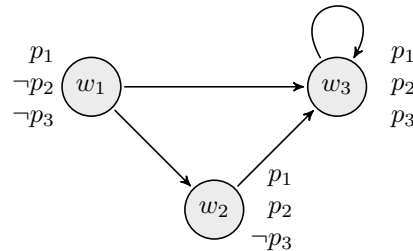
On the other hand, if $\mathfrak{M} \not\Vdash \neg\varphi$ then φ is true at some world $w \in W$. It does not follow that $\mathfrak{M}, w \Vdash \varphi$ for *every* $w \in W$. For instance, in the model of ??, $\mathfrak{M} \not\Vdash \neg p$, and also $\mathfrak{M} \not\Vdash p$.

2. Assume $\mathfrak{M} \Vdash \varphi \rightarrow \psi$ and $\mathfrak{M} \Vdash \varphi$; to show $\mathfrak{M} \Vdash \psi$ let $w \in W$ be an arbitrary world. Then $\mathfrak{M}, w \Vdash \varphi \rightarrow \psi$ and $\mathfrak{M}, w \Vdash \varphi$, so $\mathfrak{M}, w \Vdash \psi$, and since w was arbitrary, $\mathfrak{M} \Vdash \psi$.

To show that the converse fails, we need to find a model \mathfrak{M} such that $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but $\mathfrak{M} \not\Vdash \varphi \rightarrow \psi$. Consider again the model of ??: $\mathfrak{M} \not\Vdash p$ and hence (vacuously) $\mathfrak{M} \Vdash p$ only if $\mathfrak{M} \Vdash q$. However, $\mathfrak{M} \not\Vdash p \rightarrow q$, as p is true but q false at w_1 .

□

Problem syn.1. Consider the following model \mathfrak{M} for the language comprising p_1, p_2, p_3 as the only propositional variables:



Are the following **formulas** and schemas true in the model \mathfrak{M} , i.e., true at every world in \mathfrak{M} ? Explain.

1. $p \rightarrow \Diamond p$ (for p atomic);
2. $\varphi \rightarrow \Diamond \varphi$ (for φ arbitrary);

3. $\Box p \rightarrow p$ (for p atomic);
4. $\neg p \rightarrow \Diamond \Box p$ (for p atomic);
5. $\Diamond \Box \varphi$ (for φ arbitrary);
6. $\Box \Diamond p$ (for p atomic).

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Bibliography