

## syn.1 Truth in a Model

nml:syn:tru:sec Sometimes we are interested which **formulas** are true at every world in a given model. Let's introduce a notation for this.

**Definition syn.1.** A **formula**  $\varphi$  is *true in a model*  $M = \langle W, R, V \rangle$ , written  $\mathfrak{M} \Vdash \varphi$ , if and only if  $\mathfrak{M}, w \Vdash \varphi$  for every  $w \in W$ .

nml:syn:tru:prop:truthfacts **Proposition syn.2.**

1. If  $\mathfrak{M} \Vdash \varphi$  then  $\mathfrak{M} \not\Vdash \neg\varphi$ , but not vice-versa.
2. If  $\mathfrak{M} \Vdash \varphi \rightarrow \psi$  then  $\mathfrak{M} \Vdash \varphi$  only if  $\mathfrak{M} \Vdash \psi$ , but not vice-versa.

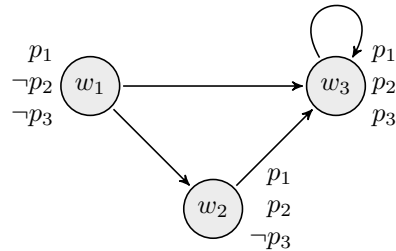
*Proof.* 1. If  $\mathfrak{M} \Vdash \varphi$  then  $\varphi$  is true at all worlds in  $W$ , and since  $W \neq \emptyset$ , it can't be that  $\mathfrak{M} \Vdash \neg\varphi$ , or else  $\varphi$  would have to be both true and false at some world.

On the other hand, if  $\mathfrak{M} \not\Vdash \neg\varphi$  then  $\varphi$  is true at some world  $w \in W$ . It does not follow that  $\mathfrak{M}, w \Vdash \varphi$  for every  $w \in W$ . For instance, in the model of ??,  $\mathfrak{M} \not\Vdash \neg p$ , and also  $\mathfrak{M} \not\Vdash p$ .

2. Assume  $\mathfrak{M} \Vdash \varphi \rightarrow \psi$  and  $\mathfrak{M} \Vdash \varphi$ ; to show  $\mathfrak{M} \Vdash \psi$  let  $w \in W$  be an arbitrary world. Then  $\mathfrak{M}, w \Vdash \varphi \rightarrow \psi$  and  $\mathfrak{M}, w \Vdash \varphi$ , so  $\mathfrak{M}, w \Vdash \psi$ , and since  $w$  was arbitrary,  $\mathfrak{M} \Vdash \psi$ .

To show that the converse fails, we need to find a model  $\mathfrak{M}$  such that  $\mathfrak{M} \Vdash \varphi$  only if  $\mathfrak{M} \Vdash \psi$ , but  $\mathfrak{M} \not\Vdash \varphi \rightarrow \psi$ . Consider again the model of ??:  $\mathfrak{M} \not\Vdash p$  and hence (vacuously)  $\mathfrak{M} \Vdash p$  only if  $\mathfrak{M} \Vdash q$ . However,  $\mathfrak{M} \not\Vdash p \rightarrow q$ , as  $p$  is true but  $q$  false at  $w_1$ .  $\square$

**Problem syn.1.** Consider the following model  $\mathfrak{M}$  for the language comprising  $p_1, p_2, p_3$  as the only propositional variables:



Are the following **formulas** and schemas true in the model  $\mathfrak{M}$ , i.e., true at every world in  $\mathfrak{M}$ ? Explain.

1.  $p \rightarrow \Diamond p$  (for  $p$  atomic);
2.  $\varphi \rightarrow \Diamond \varphi$  (for  $\varphi$  arbitrary);

3.  $\Box p \rightarrow p$  (for  $p$  atomic);
4.  $\neg p \rightarrow \Diamond \Box p$  (for  $p$  atomic);
5.  $\Diamond \Box \varphi$  (for  $\varphi$  arbitrary);
6.  $\Box \Diamond p$  (for  $p$  atomic).

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## Bibliography