Truth in a Model

Sometimes we are interested in which formulas are true at every world in a given model. Let’s introduce a notation for this.

**Definition syn.1.** A formula $\varphi$ is true in a model $M = \langle W, R, V \rangle$, written $M \models \varphi$, if and only if $M, w \models \varphi$ for every $w \in W$.

**Proposition syn.2.**

1. If $M \models \varphi$ then $M \not\models \neg \varphi$, but not vice-versa.
2. If $M \models \varphi \rightarrow \psi$ then $M \models \varphi$ only if $M \models \psi$, but not vice-versa.

**Proof.**

1. If $M \models \varphi$ then $\varphi$ is true at all worlds in $W$, and since $W \neq \emptyset$, it can’t be that $M \not\models \neg \varphi$, or else $\varphi$ would have to be both true and false at some world.

On the other hand, if $M \not\models \neg \varphi$ then $\varphi$ is true at some world $w \in W$. It does not follow that $M, w \models \varphi$ for every $w \in W$. For instance, in the model of $??$, $M \not\models \neg p$, and also $M \not\models p$.

2. Assume $M \models \varphi \rightarrow \psi$ and $M \models \varphi$; to show $M \models \psi$ let $w \in W$ be an arbitrary world. Then $M, w \models \varphi \rightarrow \psi$ and $M, w \models \psi$, so $M, w \models \psi$, and since $w$ was arbitrary, $M \models \psi$.

To show that the converse fails, we need to find a model $M$ such that $M \not\models \varphi$ only if $M \models \psi$, but $M \not\models \varphi \rightarrow \psi$. Consider again the model of $??$: $M \not\models p$ and hence (vacuously) $M \models p$ only if $M \models q$. However, $M \not\models p \rightarrow q$, as $p$ is true but $q$ false at $w_1$.

**Problem syn.1.** Consider the following model $M$ for the language comprising $p_1, p_2, p_3$ as the only propositional variables:

![Diagram of a model with worlds $w_1, w_2, w_3$ and propositions $p_1, p_2, p_3$.]

Are the following formulas and schemas true in the model $M$, i.e., true at every world in $M$? Explain.

1. $p \rightarrow \Diamond p$ (for $p$ atomic);
2. $\varphi \rightarrow \Diamond \varphi$ (for $\varphi$ arbitrary);
3. $\Box p \rightarrow p$ (for $p$ atomic);
4. $\neg p \rightarrow \Diamond \Box p$ (for $p$ atomic);
5. $\Diamond \Box \varphi$ (for $\varphi$ arbitrary);
6. $\Box \Diamond p$ (for $p$ atomic).

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Bibliography