syn.1 Truth at a World

nml:syn:trw:sec

Every modal model determines which modal formulas count as true at which worlds in it. The relation "model $\mathfrak{M}$ makes formula φ true at world w" is the basic notion of relational semantics. The relation is defined inductively and coincides with the usual characterization using truth tables for the non-modal operators.

nml:syn:trw:defn:mmodels

Definition syn.1. *Truth of a formula φ at w* in a $\mathfrak{M}$, in symbols: $\mathfrak{M}, w \Vdash \varphi$, is defined inductively as follows:

1. $\varphi \equiv \bot$: Never $\mathfrak{M}, w \Vdash \bot$.
2. $\varphi \equiv \top$: Always $\mathfrak{M}, w \Vdash \top$.
3. $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$.
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \nVdash \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \nVdash \psi$ or $\mathfrak{M}, w \Vdash \chi$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff either both $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$ or neither $\mathfrak{M}, w \Vdash \psi$ nor $\mathfrak{M}, w \Vdash \chi$.
9. (nml:syn:trw:defn:sub:mmodels-box) $\varphi \equiv \Box\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for all $w' \in W$ with Rww'.
10. (nml:syn:trw:defn:sub:mmodels-diamond) $\varphi \equiv \Diamond\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for at least one $w' \in W$ with Rww'.

Note that by clause (9), a formula $\Box\psi$ is true at w whenever there are no w' with Rww'. In such a case $\Box\psi$ is *vacuously* true at w. Also, $\Box\psi$ may be satisfied at w even if ψ is not. The truth of ψ at w does not guarantee the truth of $\Diamond\psi$ at w. This holds, however, if Rww, e.g., if R is reflexive. If there is no w' such that Rww', then $\mathfrak{M}, w \nVdash \Diamond\varphi$, for any φ.

Problem syn.1. Consider the model of **??**. Which of the following hold?

1. $\mathfrak{M}, w_1 \Vdash q$;
2. $\mathfrak{M}, w_3 \Vdash \neg q$;
3. $\mathfrak{M}, w_1 \Vdash p \vee q$;
4. $\mathfrak{M}, w_1 \Vdash \Box(p \vee q)$;
5. $\mathfrak{M}, w_3 \Vdash \Box q$;
6. $\mathfrak{M}, w_3 \Vdash \Box\bot$;
7. $\mathfrak{M}, w_1 \Vdash \Diamond q$;

8. $\mathfrak{M}, w_1 \Vdash \Box q$;
9. $\mathfrak{M}, w_1 \Vdash \neg\Box\Box\neg q$.

Proposition syn.2. nml:syn:trw:prop:dual

1. *$\mathfrak{M}, w \Vdash \Box\varphi$ iff $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$.*
2. *$\mathfrak{M}, w \Vdash \Diamond\varphi$ iff $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$.*

Proof.

1. $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$ iff $\mathfrak{M}, w \nVdash \Diamond\neg\varphi$ by definition of $\mathfrak{M}, w \Vdash$. $\mathfrak{M}, w \Vdash \Diamond\neg\varphi$ iff for some w' with Rww', $\mathfrak{M}, w' \Vdash \neg\varphi$. Hence, $\mathfrak{M}, w \nVdash \Diamond\neg\varphi$ iff for all w' with Rww', $\mathfrak{M}, w' \nVdash \neg\varphi$. We also have $\mathfrak{M}, w' \nVdash \neg\varphi$ iff $\mathfrak{M}, w' \Vdash \varphi$. Together we have $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$ iff for all w' with Rww', $\mathfrak{M}, w' \Vdash \varphi$. Again by definition of $\mathfrak{M}, w \Vdash$, that is the case iff $\mathfrak{M}, w \Vdash \Box\varphi$.
2. $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$ iff $\mathfrak{M} \nVdash \Box\neg\varphi$. $\mathfrak{M}, w \Vdash \Box\neg\varphi$ iff for all w' with Rww', $\mathfrak{M}, w' \Vdash \neg\varphi$. Hence, $\mathfrak{M}, w \nVdash \Box\neg\varphi$ iff for some w' with Rww', $\mathfrak{M}, w' \nVdash \neg\varphi$. We also have $\mathfrak{M}, w' \nVdash \neg\varphi$ iff $\mathfrak{M}, w' \Vdash \varphi$. Together we have $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$ iff for some w' with Rww', $\mathfrak{M}, w' \Vdash \varphi$. Again by definition of $\mathfrak{M}, w \Vdash$, that is the case iff $\mathfrak{M}, w \Vdash \Diamond\varphi$. □

Problem syn.2. Complete the proof of Proposition syn.2.

Problem syn.3. Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model, and suppose $w_1, w_2 \in W$ are such that:

1. $w_1 \in V(p)$ if and only if $w_2 \in V(p)$; and
2. for all $w \in W$: Rw_1w if and only if Rw_2w.

Using induction on formulas, show that for all formulas φ: $\mathfrak{M}, w_1 \Vdash \varphi$ if and only if $\mathfrak{M}, w_2 \Vdash \varphi$.

Problem syn.4. Let $\mathfrak{M} = \langle W, R, V\rangle$. Show that $\mathfrak{M}, w \Vdash \neg\Diamond\varphi$ if and only if $\mathfrak{M}, w \Vdash \Box\neg\varphi$.

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Bibliography