syn.1 Truth at a World

Every modal model determines which modal formulas count as true at which worlds in it. The relation “model $\mathcal{M}$ makes formula $\varphi$ true at world $w$” is the basic notion of relational semantics. The relation is defined inductively and coincides with the usual characterization using truth tables for the non-modal operators.

**Definition syn.1.** *Truth of a formula* $\varphi$ *at* $w$ *in a* $\mathcal{M}$, *in symbols:* $\mathcal{M}, w \models \varphi$, *is defined inductively as follows:*

1. $\varphi \equiv \bot$: Never $\mathcal{M}, w \models \bot$.
2. $\varphi \equiv \top$: Always $\mathcal{M}, w \models \top$.
3. $\mathcal{M}, w \models p$ iff $w \in V(p)$
4. $\varphi \equiv \neg \psi$: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \not\models \psi$.
5. $\varphi \equiv (\psi \land \chi)$: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \models \psi$ and $\mathcal{M}, w \models \chi$.
6. $\varphi \equiv (\psi \lor \chi)$: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \models \psi$ or $\mathcal{M}, w \models \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \not\models \psi$ or $\mathcal{M}, w \models \chi$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathcal{M}, w \models \varphi$ iff either both $\mathcal{M}, w \models \psi$ and $\mathcal{M}, w \models \chi$ or neither $\mathcal{M}, w \models \psi$ nor $\mathcal{M}, w \models \chi$.
9. $\varphi \equiv \Box \psi$: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w' \models \psi$ for all $w' \in W$ with $Rww'$
10. $\varphi \equiv \Diamond \psi$: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w' \models \psi$ for at least one $w' \in W$ with $Rww'$

Note that by clause (9), a formula $\Box \psi$ is true at $w$ whenever there are no $w'$ with $Rww'$. In such a case $\Box \psi$ is *vacuously* true at $w$. Also, $\Box \psi$ may be satisfied at $w$ even if $\psi$ is not. The truth of $\psi$ at $w$ does not guarantee the truth of $\Diamond \psi$ at $w$. This holds, however, if $Rww$, e.g., if $R$ is reflexive. If there is no $w'$ such that $Rww'$, then $\mathcal{M}, w \not\models \Diamond \varphi$, for any $\varphi$.

**Problem syn.1.** Consider the model of ???. Which of the following hold?

1. $\mathcal{M}, w_1 \models q$;
2. $\mathcal{M}, w_3 \models \neg q$;
3. $\mathcal{M}, w_1 \models p \lor q$;
4. $\mathcal{M}, w_1 \models \Box (p \lor q)$;
5. $\mathcal{M}, w_3 \models \Box q$;
6. $\mathcal{M}, w_3 \models \Box \bot$;
7. $\mathcal{M}, w_1 \models \Diamond q$;
8. \( M, w_1 \models \Box q \);
9. \( M, w_1 \models \neg \Box \neg q \).

**Proposition syn.2.**

1. \( M, w \models \Box \varphi \iff M, w \models \neg \Diamond \neg \varphi \).
2. \( M, w \models \Diamond \varphi \iff M, w \models \neg \Box \neg \varphi \).

**Proof.**

1. \( M, w \models \neg \Diamond \neg \varphi \iff M \not\models \Diamond \neg \varphi \) by definition of \( M, w \models \varphi \). Hence, \( M, w \models \Diamond \neg \varphi \iff \) for some \( w' \) with \( Rww' \), \( M, w' \models \neg \varphi \). We also have \( M, w' \not\models \neg \varphi \) iff \( M, w' \models \varphi \). Together we have \( M, w \models \Diamond \neg \varphi \iff \) for all \( w' \) with \( Rww' \), \( M, w' \models \neg \varphi \). Again by definition of \( M, w \models \varphi \), that is the case iff \( M, w \models \Box \varphi \).

2. \( M, w \models \neg \Box \neg \varphi \iff M \not\models \Box \neg \varphi \). \( M, w \models \Box \neg \varphi \iff \) for all \( w' \) with \( Rww' \), \( M, w' \models \neg \varphi \). Hence, \( M, w \models \Box \neg \varphi \iff \) for some \( w' \) with \( Rww' \), \( M, w' \models \neg \varphi \). We also have \( M, w' \not\models \neg \varphi \) iff \( M, w' \models \varphi \). Together we have \( M, w \models \neg \Box \neg \varphi \iff \) for some \( w' \) with \( Rww' \), \( M, w' \models \varphi \). Again by definition of \( M, w \models \varphi \), that is the case iff \( M, w \models \Diamond \varphi \).

\( \square \)

**Problem syn.2.** Complete the proof of Proposition syn.2.

**Problem syn.3.** Let \( M = \langle W, R, V \rangle \) be a model, and suppose \( w_1, w_2 \in W \) are such that:

1. \( w_1 \in V(p) \) if and only if \( w_2 \in V(p) \); and
2. for all \( w \in W \): \( R w_{1 \sigma} w \) if and only if \( R w_{2 \sigma} w \).

Using induction on formulas, show that for all formulas \( \varphi \): \( M, w_1 \models \varphi \) if and only if \( M, w_2 \models \varphi \).

**Problem syn.4.** Let \( M = \langle M, R, V \rangle \). Show that \( M, w \models \neg \Diamond \varphi \) if and only if \( M, w \models \Box \neg \varphi \).

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**Bibliography**