## syn.1 Truth at a World

<sup>nml:syn:trw:</sup> Every modal model determines which modal formulas count as true at which <sup>sec</sup> worlds in it. The relation "model  $\mathfrak{M}$  makes formula  $\varphi$  true at world w" is the basic notion of relational semantics. The relation is defined inductively and coincides with the usual characterization using truth tables for the non-modal operators.

nml:syn:trw: **Definition syn.1.** Truth of a formula  $\varphi$  at w in a  $\mathfrak{M}$ , in symbols:  $\mathfrak{M}, w \Vdash \varphi$ , defined inductively as follows:

- 1.  $\varphi \equiv \bot$ : Never  $\mathfrak{M}, w \Vdash \bot$ .
- 2.  $\varphi \equiv \top$ : Always  $\mathfrak{M}, w \Vdash \top$ .
- 3.  $\mathfrak{M}, w \Vdash p$  iff  $w \in V(p)$ .
- 4.  $\varphi \equiv \neg \psi$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}, w \nvDash \psi$ .
- 5.  $\varphi \equiv (\psi \land \chi)$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}, w \Vdash \psi$  and  $\mathfrak{M}, w \Vdash \chi$ .
- 6.  $\varphi \equiv (\psi \lor \chi)$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}, w \Vdash \psi$  or  $\mathfrak{M}, w \Vdash \chi$  (or both).
- 7.  $\varphi \equiv (\psi \to \chi)$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}, w \nvDash \psi$  or  $\mathfrak{M}, w \Vdash \chi$ .
- 8.  $\varphi \equiv (\psi \leftrightarrow \chi)$ :  $\mathfrak{M}, w \Vdash \varphi$  iff either both  $\mathfrak{M}, w \Vdash \psi$  and  $\mathfrak{M}, w \Vdash \chi$  or neither  $\mathfrak{M}, w \Vdash \psi$  nor  $\mathfrak{M}, w \Vdash \chi$ .
- 9.  $\varphi \equiv \Box \psi$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}, w' \Vdash \psi$  for all  $w' \in W$  with Rww'.
- 10.  $\varphi \equiv \Diamond \psi$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}, w' \Vdash \psi$  for at least one  $w' \in W$  with Rww'.

 $\begin{array}{ll} & \text{nml:syn:trw:} & 10. \quad \varphi \equiv \Diamond \psi \\ & \text{defn:sub:mmodels-diamond} \\ & \text{Note that} \\ & \text{no } \text{ uv} \text{ with } P \end{array}$ 

nml:syn:trw: defn:sub:mmodels-box

> Note that by clause (9), a formula  $\Box \psi$  is true at w whenever there are no w' with Rww'. In such a case  $\Box \psi$  is *vacuously* true at w. Also,  $\Box \psi$  may be satisfied at w even if  $\psi$  is not. The truth of  $\psi$  at w does not guarantee the truth of  $\Diamond \psi$  at w. This holds, however, if Rww, e.g., if R is reflexive. If there is no w' such that Rww', then  $\mathfrak{M}, w \nvDash \Diamond \varphi$ , for any  $\varphi$ .

**Problem syn.1.** Consider the model of **??**. Which of the following hold?

- 1.  $\mathfrak{M}, w_1 \Vdash q;$
- 2.  $\mathfrak{M}, w_3 \Vdash \neg q;$
- 3.  $\mathfrak{M}, w_1 \Vdash p \lor q;$
- 4.  $\mathfrak{M}, w_1 \Vdash \Box (p \lor q);$
- 5.  $\mathfrak{M}, w_3 \Vdash \Box q;$
- 6.  $\mathfrak{M}, w_3 \Vdash \Box \bot;$
- 7.  $\mathfrak{M}, w_1 \Vdash \Diamond q;$

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8.  $\mathfrak{M}, w_1 \Vdash \Box q;$ 

9.  $\mathfrak{M}, w_1 \Vdash \neg \Box \Box \neg q$ .

## Proposition syn.2.

nml:syn:trw: prop:dual

1.  $\mathfrak{M}, w \Vdash \Box \varphi$  iff  $\mathfrak{M}, w \Vdash \neg \Diamond \neg \varphi$ .

2.  $\mathfrak{M}, w \Vdash \Diamond \varphi \text{ iff } \mathfrak{M}, w \Vdash \neg \Box \neg \varphi.$ 

- *Proof.* 1.  $\mathfrak{M}, w \Vdash \neg \Diamond \neg \varphi$  iff  $\mathfrak{M}, w \nvDash \Diamond \neg \varphi$  by definition of  $\mathfrak{M}, w \Vdash \mathfrak{M}, w \Vdash \Diamond \neg \varphi$  iff for some w' with  $Rww', \mathfrak{M}, w' \Vdash \neg \varphi$ . Hence,  $\mathfrak{M}, w \nvDash \Diamond \neg \varphi$  iff for all w' with  $Rww', \mathfrak{M}, w' \nvDash \neg \varphi$ . We also have  $\mathfrak{M}, w' \nvDash \neg \varphi$  iff  $\mathfrak{M}, w' \Vdash \varphi$ . Together we have  $\mathfrak{M}, w \Vdash \neg \Diamond \neg \varphi$  iff for all w' with  $Rww', \mathfrak{M}, w' \Vdash \varphi$ . Again by definition of  $\mathfrak{M}, w \Vdash$ , that is the case iff  $\mathfrak{M}, w \Vdash \Box \varphi$ .
  - 2.  $\mathfrak{M}, w \Vdash \neg \Box \neg \varphi$  iff  $\mathfrak{M} \nvDash \Box \neg \varphi$ .  $\mathfrak{M}, w \Vdash \Box \neg \varphi$  iff for all w' with Rww',  $\mathfrak{M}, w' \Vdash \neg \varphi$ . Hence,  $\mathfrak{M}, w \nvDash \Box \neg \varphi$  iff for some w' with  $Rww', \mathfrak{M}, w' \nvDash \neg \varphi$ . We also have  $\mathfrak{M}, w' \nvDash \neg \varphi$  iff  $\mathfrak{M}, w' \Vdash \varphi$ . Together we have  $\mathfrak{M}, w \Vdash \neg \Box \neg \varphi$ iff for some w' with  $Rww', \mathfrak{M}, w' \Vdash \varphi$ . Again by definition of  $\mathfrak{M}, w \Vdash$ , that is the case iff  $\mathfrak{M}, w \Vdash \Diamond \varphi$ .

Problem syn.2. Complete the proof of Proposition syn.2.

**Problem syn.3.** Let  $\mathfrak{M} = \langle W, R, V \rangle$  be a model, and suppose  $w_1, w_2 \in W$  are such that:

- 1.  $w_1 \in V(p)$  if and only if  $w_2 \in V(p)$ ; and
- 2. for all  $w \in W$ :  $Rw_1w$  if and only if  $Rw_2w$ .

Using induction on formulas, show that for all formulas  $\varphi \colon \mathfrak{M}, w_1 \Vdash \varphi$  if and only if  $\mathfrak{M}, w_2 \Vdash \varphi$ .

**Problem syn.4.** Let  $\mathfrak{M} = \langle W, R, V \rangle$ . Show that  $\mathfrak{M}, w \Vdash \neg \Diamond \varphi$  if and only if  $\mathfrak{M}, w \Vdash \Box \neg \varphi$ .

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Bibliography