

## syn.1 Tautological Instances

nml:syn:tau:  
sec

A modal-free formula is a tautology if it is true under every truth-value assignment. Clearly, every tautology is true at every world in every model. But for formulas involving  $\Box$  and  $\Diamond$ , the notion of tautology is not defined. Is it the case, e.g., that  $\Box p \vee \neg \Box p$ —an instance of the principle of excluded middle—is valid? The notion of a *tautological instance* helps: a formula that is a substitution instance of a (non-modal) tautology. It is not surprising, but still requires proof, that every tautological instance is valid.

explanation

**Definition syn.1.** A modal formula  $\psi$  is a *tautological instance* if and only if there is a modal-free tautology  $\varphi$  with propositional variables  $p_1, \dots, p_n$  and formulas  $\theta_1, \dots, \theta_n$  such that  $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ .

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lem:valid-taut

**Lemma syn.2.** Suppose  $\varphi$  is a modal-free formula whose propositional variables are  $p_1, \dots, p_n$ , and let  $\theta_1, \dots, \theta_n$  be modal formulas. Then for any assignment  $\mathbf{v}$ , any model  $\mathfrak{M} = \langle W, R, V \rangle$ , and any  $w \in W$  such that  $\mathbf{v}(p_i) = \mathbb{T}$  if and only if  $\mathfrak{M}, w \Vdash \theta_i$  we have that  $\mathbf{v} \models \varphi$  if and only if  $\mathfrak{M}, w \Vdash \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ .

*Proof.* By induction on  $\varphi$ .

1.  $\varphi \equiv \perp$ : Both  $\mathbf{v} \not\models \perp$  and  $\mathfrak{M}, w \not\Vdash \perp$ .
2.  $\varphi \equiv \top$ : Both  $\mathbf{v} \models \top$  and  $\mathfrak{M}, w \Vdash \top$ .
3.  $\varphi \equiv p_i$ :

$$\begin{aligned} \mathbf{v} \models p_i &\Leftrightarrow \mathbf{v}(p_i) = \mathbb{T} \\ &\text{by definition of } \mathbf{v} \models p_i \\ &\Leftrightarrow \mathfrak{M}, w \Vdash \theta_i \\ &\text{by assumption} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash p_i[\theta_1/p_1, \dots, \theta_n/p_n] \\ &\text{since } p_i[\theta_1/p_1, \dots, \theta_n/p_n] \equiv \theta_i. \end{aligned}$$

4.  $\varphi \equiv \neg\psi$ :

$$\begin{aligned} \mathbf{v} \models \neg\psi &\Leftrightarrow \mathbf{v} \not\models \psi \\ &\text{by definition of } \mathbf{v} \models; \\ &\Leftrightarrow \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \\ &\text{by induction hypothesis} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash \neg\psi[\theta_1/p_1, \dots, \theta_n/p_n] \\ &\text{by definition of } \mathbf{v} \models. \end{aligned}$$

5.  $\varphi \equiv (\psi \wedge \chi)$ :

$$\begin{aligned}
\mathbf{v} \models \psi \wedge \chi &\Leftrightarrow \mathbf{v} \models \psi \text{ and } \mathbf{v} \models \chi \\
&\text{by definition of } \mathbf{v} \models \\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \wedge \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

6.  $\varphi \equiv (\psi \vee \chi)$ :

$$\begin{aligned}
\mathbf{v} \models \psi \vee \chi &\Leftrightarrow \mathbf{v} \models \psi \text{ or } \mathbf{v} \models \chi \\
&\text{by definition of } \mathbf{v} \models; \\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \vee \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

7.  $\varphi \equiv (\psi \rightarrow \chi)$ :

$$\begin{aligned}
\mathbf{v} \models \psi \rightarrow \chi &\Leftrightarrow \mathbf{v} \not\models \psi \text{ or } \mathbf{v} \models \chi \\
&\text{by definition of } \mathbf{v} \models \\
&\Leftrightarrow \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \rightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

8.  $\varphi \equiv (\psi \leftrightarrow \chi)$ :

$$\begin{aligned}
\mathbf{v} \models \psi \leftrightarrow \chi &\Leftrightarrow \text{either } \mathbf{v} \models \psi \text{ and } \mathbf{v} \models \chi \\
&\quad \text{or } \mathbf{v} \not\models \psi \text{ and } \mathbf{v} \not\models \chi \\
&\text{by definition of } \mathbf{v} \models \\
&\Leftrightarrow \text{either } \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\quad \text{or } \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \not\Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \leftrightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash. \quad \square
\end{aligned}$$

*nml:syn:tau:* **Proposition syn.3.** *All tautological instances are valid.*  
*prop:valid-taut*

*Proof.* Contrapositively, suppose  $\varphi$  is such that  $\mathfrak{M}, w \not\models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ , for some model  $\mathfrak{M}$  and world  $w$ . Define an assignment  $\mathfrak{v}$  such that  $\mathfrak{v}(p_i) = \mathbb{T}$  if and only if  $\mathfrak{M}, w \models \theta_i$  (and  $\mathfrak{v}$  assigns arbitrary values to  $q \notin \{p_1, \dots, p_n\}$ ). Then by **Lemma syn.2**,  $\mathfrak{v} \not\models \varphi$ , so  $\varphi$  is not a tautology.  $\square$

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## Bibliography