syn.1 Tautological Instances

mod:syn:tau:sec

Definition syn.1. A modal formula ψ is a *tautological instance* if and only if there is a modal-free tautology φ and formulas $\theta_1, \ldots, \theta_n$ such that $\psi = \varphi[\theta_1/p_1, \ldots, \theta_n/p_n/]$.

mod:syn:tau:lem:valid-taut **Lemma syn.2.** *Suppose φ is a modal-free formula all of whose propositional variables are among $p_1, \ldots, p_n$, and let $\theta_1, \ldots, \theta_n$ be modal formulas. Then for any assignment v, any model $\mathfrak{M} = \langle W, R, V \rangle$, and any $w \in W$ such that $v(p_i) = 1$ if and only if $\mathfrak{M}, w \models \theta_i$ we have that $v \models \varphi$ if and only if $\mathfrak{M}, w \models \varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$.*

Proof. By induction on φ.

1. φ is atomic: then by the hypothesis it must be some p_i, whence:

$$\begin{aligned} v \models p_i \Leftrightarrow v(p_i) = 1 &\Leftrightarrow \mathfrak{M}, w \models \theta_i \\ &\Leftrightarrow \mathfrak{M}, w \models \varphi[\theta_1/p_1, \ldots, \theta_n/p_n]. \end{aligned}$$

2. $\varphi \equiv \neg\psi$:

$$\begin{aligned} v \models \neg\psi &\Leftrightarrow v \not\models \psi && \text{by definition of } v \models \neg\psi; \\ &\Leftrightarrow \mathfrak{M}, w \not\models \psi && \text{by induction hypothesis;} \\ &\Leftrightarrow \mathfrak{M}, w \models \neg\psi && \text{by definition of } v \models \neg\psi. \end{aligned}$$

3. $\varphi \equiv \psi \to \chi$:

$$\begin{aligned} v \models \psi \to \chi &\Leftrightarrow v \not\models \psi \text{ or } v \models \chi && \text{by definition of } v \models \psi \to \chi; \\ &\Leftrightarrow \mathfrak{M}, w \not\models \psi \text{ or } \mathfrak{M}, w \models \chi, && \text{by induction hypothesis;} \\ &\Leftrightarrow \mathfrak{M}, w \models \psi \to \chi && \text{by definition of } \mathfrak{M}, w \models \psi \to \chi. \end{aligned}$$ □

mod:syn:tau:thm:valid-taut **Theorem syn.3.** *All tautological instances are valid.*

Proof. Contrapositively, suppose φ is such that $\mathfrak{M}, w \not\models \varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$, for some model $\mathfrak{M}$ and world w. Define an assignment v such that $v(p_i) = 1$ if and only if $\mathfrak{M}, w \models \theta_i$ (and v assigns arbitrary values to $q \notin \{p_1, \ldots, p_n\}$). Then by Lemma syn.2, $v \not\models \varphi$, so φ is not a tautology. □

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Bibliography