syn.1 Tautological Instances

mod:syn:tau:sec

A modal-free formula is a tautology if it is true under every truth-value assignment. Clearly, every tautology is true at every world in every model. But for formulas involving $\Box$ and $\Diamond$, the notion of tautology is not defined. Is it the case, e.g., that $\Box p \lor \lnot\Box p$—an instance of the principle of excluded middle—is valid? The notion of a *tautological instance* helps: a formula that is a substitution instance of a (non-modal) tautology. It is not surprising, but still requires proof, that every tautological instance is valid. explanation

Definition syn.1. A modal formula ψ is a *tautological instance* if and only if there is a modal-free tautology φ with propositional variables $p_1, \dots, p_n$ and formulas $\theta_1, \dots, \theta_n$ such that $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.

mod:syn:tau:lem:valid-taut

Lemma syn.2. *Suppose φ is a modal-free formula whose propositional variables are $p_1, \dots, p_n$, and let $\theta_1, \dots, \theta_n$ be modal formulas. Then for any assignment $\mathfrak{v}$, any model $\mathfrak{M} = \langle W, R, V\rangle$, and any $w \in W$ such that $\mathfrak{v}(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \Vdash \theta_i$ we have that $\mathfrak{v} \vDash \varphi$ if and only if $\mathfrak{M}, w \Vdash \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.*

Proof. By induction on φ.

1. $\varphi \equiv \bot$: Both $\mathfrak{v} \nvDash \bot$ and $\mathfrak{M}, w \nVdash \bot$.

2. $\varphi \equiv \top$: Both $\mathfrak{v} \vDash \top$ and $\mathfrak{M}, w \Vdash \top$.

3. $\varphi \equiv p_i$:

$$\begin{aligned}
\mathfrak{v} \vDash p_i &\Leftrightarrow \mathfrak{v}(p_i) = \mathbb{T} && \text{by definition of } \mathfrak{v} \vDash p_i;\\
&\Leftrightarrow \mathfrak{M}, w \Vdash \theta_i && \text{by assumption}\\
&\Leftrightarrow \mathfrak{M}, w \Vdash p_i[\theta_1/p_1, \dots, \theta_n/p_n] && \text{since } p_i[\theta_1/p_1, \dots, \theta_n/p_n] \equiv \theta_i.
\end{aligned}$$

4. $\varphi \equiv \lnot\psi$:

$$\begin{aligned}
\mathfrak{v} \vDash \lnot\psi &\Leftrightarrow \mathfrak{v} \nvDash \psi && \text{by definition of } \mathfrak{v} \vDash;\\
&\Leftrightarrow \mathfrak{M}, w \nVdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by induction hypothesis;}\\
&\Leftrightarrow \mathfrak{M}, w \Vdash \lnot\psi[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{v} \vDash.
\end{aligned}$$

5. $\varphi \equiv (\psi \land \chi)$:

$$\begin{aligned}
\mathfrak{v} \vDash \psi \land \chi &\Leftrightarrow \mathfrak{v} \vDash \psi \text{ and } \mathfrak{v} \vDash \chi && \text{by definition of } \mathfrak{v} \vDash;\\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} &&\\
&\qquad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n], && \text{by induction hypothesis;}\\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \land \chi)[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

6. $\varphi \equiv (\psi \lor \chi)$:

$$
\begin{aligned}
\mathfrak{v} \models \psi \lor \chi &\Leftrightarrow \mathfrak{v} \models \psi \text{ or } \mathfrak{v} \models \chi && \text{by definition of } \mathfrak{v} \models;\\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or}\\
&\ \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n], && \text{by induction hypothesis;}\\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \lor \chi)[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}
$$

7. $\varphi \equiv (\psi \rightarrow \chi)$:

$$
\begin{aligned}
\mathfrak{v} \models \psi \rightarrow \chi &\Leftrightarrow \mathfrak{v} \not\models \psi \text{ or } \mathfrak{v} \models \chi && \text{by definition of } \mathfrak{v} \models;\\
&\Leftrightarrow \mathfrak{M}, w \nVdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or}\\
&\ \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n], && \text{by induction hypothesis;}\\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \rightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}
$$

8. $\varphi \equiv (\psi \leftrightarrow \chi)$:

$$
\begin{aligned}
\mathfrak{v} \models \psi \rightarrow \chi &\Leftrightarrow \text{either } \mathfrak{v} \models \psi \text{ and } \mathfrak{v} \models \chi\\
&\ \text{or } \mathfrak{v} \not\models \psi \text{ and } \mathfrak{v} \not\models \chi && \text{by definition of } \mathfrak{v} \models;\\
&\Leftrightarrow \text{either } \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and}\\
&\ \qquad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n]\\
&\ \text{or } \mathfrak{M}, w \nVdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and}\\
&\ \qquad \mathfrak{M}, w \nVdash \chi[\theta_1/p_1, \dots, \theta_n/p_n], && \text{by induction hypothesis;}\\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \leftrightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}
$$

□

Proposition syn.3. *All tautological instances are valid.* *mod:syn:tau: prop:valid-taut*

Proof. Contrapositively, suppose φ is such that $\mathfrak{M}, w \nVdash \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$, for some model $\mathfrak{M}$ and world w. Define an assignment v such that $v(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \Vdash \theta_i$ (and v assigns arbitrary values to $q \notin \{p_1, \dots, p_n\}$). Then by Lemma syn.2, $\mathfrak{v} \not\models \varphi$, so φ is not a tautology. □

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Bibliography