Chapter udf

Syntax and Semantics of Normal Modal Logics

syn.1 Introduction

mod:syn:int:sec

Modal Logic deals with *modal propositions* and the entailment relations among them. Examples of modal propositions are the following:

1. It is necessary that $2 + 2 = 4$.
2. It is necessarily possible that it will rain tomorrow.
3. If it is necessarily possible that φ then it is possible that φ.

Possibility and necessity are not the only modalities: other unary connectives are also classified as modalities, for instance, "it ought to be the case that φ," "It will be the case that φ," "Dana knows that φ," or "Dana believes that φ."

Modal logic makes its first appearance in Aristotle's *De Interpretatione*: he was the first to notice that necessity implies possibility, but not vice versa; that possibility and necessity are inter-definable; that If $\varphi \wedge \psi$ is possibly true then φ is possibly true and ψ is possibly true, but not conversely; and that if $\varphi \rightarrow \psi$ is necessary, then if φ is necessary, so is ψ.

The first modern approach to modal logic was the work of C. I. Lewis, culminating with Lewis and Langford, *Symbolic Logic* (1932). Lewis & Langford were unhappy with the representation of implication by means of the material conditional: $\varphi \rightarrow \psi$ is a poor substitute for "φ implies ψ." Instead, they proposed to characterize implication as "Necessarily, if φ then ψ," symbolized as $\varphi \strictif \psi$. In trying to sort out the different properties, Lewis indentified five different modal systems, **S1**, ..., **S4**, **S5**, the last two of which are still in use.

The approach of Lewis and Langford was purely *syntactical*: they identified reasonable axioms and rules and investigated wat was provable with those means. A semantic approach remained elusive for a long time, until a first attempt was made by Rudolf Carnap in *Meaning and Necessity* (1947) using the notion of a *state description*, i.e., a collection of atomic sentences (those that are "true" in that state description). After lifting the truth definition to arbitrary sentences φ, Carnap defines φ to be *necessarily true* if it is true in all state descriptions. Carnap's approach could not handle *iterated* modalities, in that sentences of the form "Possibly necessarily ... possibly φ" always reduce to the innermost modality.

The major breakthrough in modal semantics came with Saul Kripke's article "A Completeness Theorem in Modal Logic" (JSL 1959). Kripke based his work on Leibniz's idea that a statement is necessarily true if it is true "at all possible worlds." This idea, though, suffers from the same drawbacks as Carnap's, in that the truth of statement at a world w (or a state description s) does not depend on w at all. So Kripke assumed that worlds are related by an *accessibility relation* R, and that a statement of the form "Necessarily φ" is true at a world w if and only if φ is true at all worlds w' *accessible from* w. Semantics that provide some version of this approach are called Kripke semantics and made possible the tumultuous development of modal logics (in the plural).

When interpreted by the Kripke semantics, modal logic shows us what *relational structures* look like "from the inside." A relational structure is just a set equipped with a binary relation (for instance, the set of students in the class ordered by their social security number is a relational structure). But in fact relational structures come in all sorts of domains: besides relative possibility of states of the world, we can have epistemic states of some agent related by epistemic possibility, or states of a dynamical system with their state transitions, etc. Modal logic can be used to model all of these: the first give us ordinary, alethic, modal logic; the others give us epistemic logic, dynamic logic, etc.

We focus on one particular angle, known to modal logicians as "correspondence theory." One of the most significant early discoveries of Kripke's is that many properties of the accessibility relation R (whether it is transitive, symmetric, etc.) can be characterized *in the modal language* itself by means of appropriate "modal schemas." Modal logicians say, for instance, that the reflexivity of R "corresponds" to the schema "If necessarily φ, then φ". We explore mainly the correspondence theory of a number of classical systems of modal logic (e.g., **S4** and **S5**) obtained by a combination of the schemas D, T, B, 4, and 5.

syn.2 The Language of Basic Modal Logic

The basic language of modal logic contains a set Var of propositional variables p_1, p_2, ..., the familiar logical connectives $\neg$ ("not"), $\wedge$ ("and"), $\vee$ ("or"), $\rightarrow$, ("if ... then"), the symbols $\top$ (the truth symbol) and $\bot$ (the falsity symbol), as well as the two basic modalities $\Box$ and $\Diamond$. mod:syn:lan:sec

Definition syn.1. *Formulas* of the basic modal language are inductively defined as follows:

1. Every propositional variable p_i is an (atomic) formula.
2. $\top$ is an (atomic) formula
3. $\bot$ is an (atomic) formula.
4. If φ is a formula, so is $\neg\varphi$.
5. If φ and ψ are formulas, so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$.
6. If φ is a formula, so is $\Box\varphi$.
7. Nothing else is a formula.

If a formula φ does not contain $\Box$, we say it is *modal-free*.

$\Diamond A$ abbreviates $\neg\Box\neg\varphi$. So for instance, $\Diamond\Box p \rightarrow \Diamond\Diamond p$ is short for $\neg\Box\neg\Box p \rightarrow \neg\Box\neg\neg\Box\neg p$.

syn.3 Simultaneous Substitution

mod:syn:sub:sec

Definition syn.2. Where φ is a modal formula all of whose propositional variables are among p_1, …, p_n, and χ_1, …, χ_n are also modal formulas, we define $\varphi[\chi_1/p_1, \ldots, \chi_n/p_n]$ as the result of simultaneously substituting each χ_i for p_i in A. Formally, this is a definition by induction on φ:

1. If φ is the propositional variable q, then $\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = \chi_i$ if Q is p_i for some $i \leq n$, and $\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = Q$ otherwise.
2. If $A = \neg\psi$, then $\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = \neg\psi[\chi_1/p_1, \ldots, \chi_n/p_n]$.
3. If $\varphi = (\psi \wedge \theta)$, then
$$\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = (\psi[\chi_1/p_1, \ldots, \chi_n/p_n] \wedge \theta[\chi_1/p_1, \ldots, \chi_n/p_n]).$$
4. If $\varphi = (\psi \vee \theta)$, then
$$\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = (\psi[\chi_1/p_1, \ldots, \chi_n/p_n] \vee \theta[\chi_1/p_1, \ldots, \chi_n/p_n]).$$
5. If $\varphi = (\psi \rightarrow \theta)$, then
$$\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = (\psi[\chi_1/p_1, \ldots, \chi_n/p_n] \rightarrow \theta[\chi_1/p_1, \ldots, \chi_n/p_n]).$$
6. If $\varphi = (\psi \leftrightarrow \theta)$, then
$$\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = (\psi[\chi_1/p_1, \ldots, \chi_n/p_n] \leftrightarrow \theta[\chi_1/p_1, \ldots, \chi_n/p_n]).$$
7. If $\varphi = \Box\psi$, then $\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = \Box\psi[\chi_1/p_1, \ldots, \chi_n/p_n]$.
8. If $\varphi = \Diamond\psi$, then $\varphi[\chi_1/p_1, \ldots, \chi_n/p_n] = \Diamond\psi[\chi_1/p_1, \ldots, \chi_n/p_n]$.

The formula $\varphi[\chi_1/p_1, \ldots, \chi_n/p_n]$ is called a *substitution instance* of φ.

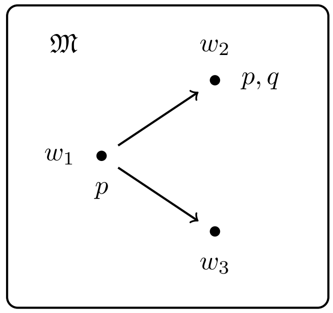


Figure syn.1: A simple model.

mod:syn:rel: fig:simple

syn.4 Relational Models

mod:syn:rel: sec

Definition syn.3. A *model* for the basic modal language is a triple $\mathfrak{M} = \langle W, R, V \rangle$, where

1. W is a nonempty set of "worlds,"
2. R is a binary accessibility relation on W, and
3. V is a function assigning to each propositional variable p a set $V(p)$ of possible worlds.

The great advantage of relational semantics is that models can be represented by means of simple diagrams, such as the one in Figure syn.1. Worlds are represented by nodes, and world w' is accessible from w precisely when there is an arrow from w to w'. Moreover, we write p next to a world precisely when $w \in V(p)$.

syn.5 Truth at a World

mod:syn:trw: sec

Definition syn.4. *Truth of a formula φ at w* in a $\mathfrak{M}$, $\mathfrak{M}, w \models \varphi$, is defined inductively as follows: mod:syn:trw: defn:mmodels

1. $\mathfrak{M}, w \models p$ iff $w \in V(p)$
2. $\mathfrak{M}, w \models \top$
3. $\mathfrak{M}, w \not\models \bot$
4. $\mathfrak{M}, w \models \neg\psi$ iff $\mathfrak{M}, w \not\models \psi$
5. $\mathfrak{M}, w \models \varphi \land \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

6. $\mathfrak{M}, w \models \varphi \lor \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$ (or both)

7. $\mathfrak{M}, w \models \varphi \to \psi$ iff $\mathfrak{M}, w \not\models \varphi$ or $\mathfrak{M}, w \models \psi$

mod:syn:trw: defn:sub:mmodels-box 8. $\mathfrak{M}, w \models \Box\varphi$ iff $\mathfrak{M}, w' \models \varphi$ for all $w' \in W$ with wRw'

Note that by clause (8), a formula $\Box\psi$ is satisfied at w whenever there are no w' with wRw'. In such a case $\Box\psi$ is *vacuously* satisfied at w. Also, $\Box\psi$ may be satisifed at w even if ψ is not, and the truth of ψ at w does not guarantee the truth of $\Diamond\psi$ there—this holds if wRw, e.g., if R is reflexive.

Problem syn.1. Consider the model of Figure syn.1. Which of the following hold?

1. $\mathfrak{M}, w_1 \models q$;
2. $\mathfrak{M}, w_3 \models \neg q$;
3. $\mathfrak{M}, w_1 \models p \lor q$;
4. $\mathfrak{M}, w_1 \models \Box(p \lor q)$;
5. $\mathfrak{M}, w_3 \models \Box q$;
6. $\mathfrak{M}, w_3 \models \Box\bot$;
7. $\mathfrak{M}, w_1 \models \Diamond q$;
8. $\mathfrak{M}, w_1 \models \Box q$;
9. $\mathfrak{M}, w_1 \models \neg\Box\Box\neg q$.

Problem syn.2. Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model, and suppose $u, v \in W$ are such that:

1. $u \in V(p)$ if and only if $v \in V(p)$; and
2. for all $w \in W$: Ruw if and only if Rvw.

Using induction on formulas, show that for all formulas φ: $\mathfrak{M}, u \models \varphi$ if and only if $\mathfrak{M}, v \models \varphi$.

Problem syn.3. Let $\mathfrak{M} = \langle M, R, V\rangle$. Show that $\mathfrak{M}, w \models \Diamond\varphi$ if and only if, for some w' with Rww', $\mathfrak{M}, w' \models \varphi]$.

syn.6 Entailment

mod:syn:ent: sec

Definition syn.5. If Γ is a set of formulas and φ a formula, then Γ *entails* φ—in symbols: $\Gamma \vDash \varphi$—if and only if for every model $\mathfrak{M} = (W, R, V)$ and world $w \in W$, if $\mathfrak{M}, w \models \psi$ for every $\psi \in \Gamma$, then $\mathfrak{M}, w \models \varphi$. If Γ contains a single formula ψ, then we write $\psi \vDash \varphi$.

Problem syn.4. Show that $\Box(p \to q) \nvDash p \to \Box q$ and $p \to \Box q \nvDash \Box(p \to q)$.

syn.7 Truth in a Model

mod:syn:tru:sec

Definition syn.6. A formula φ is *true in a model* $M = \langle W, R, V \rangle$, written $\mathfrak{M} \models \varphi$, if and only if $\mathfrak{M}, w \models \varphi$ for every $w \in W$.

Proposition syn.7. *mod:syn:tru:prop:truthfacts*

1. *If* $\mathfrak{M} \models \varphi$ *then* $\mathfrak{M} \not\models \neg\varphi$, *but* not *vice-versa.*

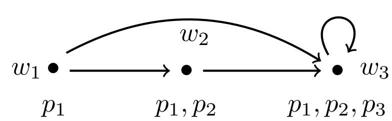
2. *If* $\mathfrak{M} \models \varphi \to \psi$ *then* $\mathfrak{M} \models \varphi$ *only if* $\mathfrak{M} \models \psi$, *but* not *vice-versa.*

Proof. 1. If $\mathfrak{M} \models \varphi$ then φ is true at all worlds in W, and since $W \neq \emptyset$, it can't be that $\mathfrak{M} \models \neg\varphi$, or else φ would have to be both true and false at some world. Conversely, if $\mathfrak{M} \not\models \neg\varphi$ then φ is true at some world $w \in W$; it does not follow that $\mathfrak{M} \models \varphi$. For instance, in the model of Figure syn.1, $\mathfrak{M} \not\models \neg p$, but it does not follow that $\mathfrak{M} \models p$.

2. Assume $\mathfrak{M} \models \varphi \to \psi$ and $\mathfrak{M} \models \varphi$; to show $\mathfrak{M} \models \psi$ let $w \in W$ be an arbitrary world. Then $\mathfrak{M}, w \models \varphi \to \psi$ and $\mathfrak{M}, w \models \psi$, so $\mathfrak{M}, w \models \varphi$, and since w was arbitrary, $\mathfrak{M} \models \varphi$. The converse fails: we need to find a model $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi$ only if $\mathfrak{M} \models \psi$, but $\mathfrak{M} \not\models \varphi \to \psi$. Consider again the model of Figure syn.1: $\mathfrak{M} \not\models p$ and hence (vacuously) $\mathfrak{M} \models p$ only if $\mathfrak{M} \models q$. However, $\mathfrak{M} \not\models p \to q$, as p is true but q false at w_1.

□

Problem syn.5. Consider the following model $\mathfrak{M}$ for the language comprising p_1, p_2, p_3 as the only propositional variables:



Are the following formulas and schemas true in the model $\mathfrak{M}$, i.e., true at every world in $\mathfrak{M}$? Explain.

1. $p \to \Diamond p$ (for p atomic);

2. $\varphi \to \Diamond\varphi$ (for φ arbitrary);

3. $\Box p \to p$ (for p atomic);

4. $\neg p \to \Diamond\Box p$ (for p atomic);

5. $\Diamond\Box\varphi$ (for φ arbitrary);

6. $\Box\Diamond p$ (for p atomic).

syn.8 Validity

mod:syn:val: sec

Definition syn.8. A formula φ is *valid* in a class $\mathcal{C}$ of models if it is true in every model in $\mathcal{C}$ (i.e., true at every world in every model in $\mathcal{C}$). If φ is valid in $\mathcal{C}$ we write $\mathcal{C} \vDash \varphi$, and we write $\vDash \varphi$ if φ is valid in the class of *all* models.

Proposition syn.9. *If φ is valid in $\mathcal{C}$ it is also valid in each class $\mathcal{C}' \subseteq \mathcal{C}$.*

mod:syn:val: prop:Nec-rule

Proposition syn.10. *If φ is valid, then so is $\Box\varphi$.*

Proof. Assume $\vDash \varphi$. To show $\vDash \Box\varphi$ let $\mathfrak{M} = \langle W, R, V\rangle$ be a model and $w \in W$. If Rww' then $\mathfrak{M}, w' \models \varphi$, since φ is valid, and so also $\mathfrak{M}, w \models \Box\varphi$. Since $\mathfrak{M}$ and w were arbitrary, $\vDash \Box\varphi$. □

Problem syn.6. Show that the following are valid:

1. $\vDash \Box p \to \Box(q \to p)$;
2. $\vDash \Box\neg\bot$;
3. $\vDash \Box p \to (\Box q \to \Box p)$.

Problem syn.7. Show that $\varphi \to \Box\varphi$ is valid in the class $\mathcal{C}$ of models $\mathfrak{M} = \langle W, R, V\rangle$ where $W = \{w\}$. Similarly, show that $\psi \to \Box\varphi$ and $\Diamond\varphi \to \psi$ are valid in the class of models $\mathfrak{M} = \langle W, R, V\rangle$ where $R = \emptyset$.

syn.9 Tautological Instances

mod:syn:tau: sec

Definition syn.11. A modal formula ψ is a *tautological instance* if and only if there is a modal-free tautology φ and formulas $\theta_1, \dots, \theta_n$ such that $\psi = \varphi[\theta_1/p_1, \dots, \theta_n/p_n/]$.

mod:syn:tau: lem:valid-taut

Lemma syn.12. *Suppose φ is a modal-free formula all of whose propositional variables are among $p_1, \dots, p_n$, and let $\theta_1, \dots, \theta_n$ be modal formulas. Then for any assignment v, any model $\mathfrak{M} = \langle W, R, V\rangle$, and any $w \in W$ such that $v(p_i) = 1$ if and only if $\mathfrak{M}, w \models \theta_i$ we have that $v \models \varphi$ if and only if $\mathfrak{M}, w \models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.*

Proof. By induction on φ.

1. φ is atomic: then by the hypothesis it must be some p_i, whence:

$$\begin{aligned} v \models p_i \Leftrightarrow v(p_i) = 1 &\Leftrightarrow \mathfrak{M}, w \models \theta_i \\ &\Leftrightarrow \mathfrak{M}, w \models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]. \end{aligned}$$

2. $\varphi \equiv \neg\psi$:

$$\begin{aligned}
v \models \neg\psi &\Leftrightarrow v \not\models \psi && \text{by definition of } v \models \neg\psi;\\
&\Leftrightarrow \mathfrak{M}, w \not\models \psi && \text{by induction hypothesis;}\\
&\Leftrightarrow \mathfrak{M}, w \models \neg\psi && \text{by definition of } v \models \neg\psi.
\end{aligned}$$

3. $\varphi \equiv \psi \to \chi$:

$$\begin{aligned}
v \models \psi \to \chi &\Leftrightarrow v \not\models \psi \text{ or } v \models \chi && \text{by definition of } v \models \psi \to \chi;\\
&\Leftrightarrow \mathfrak{M}, w \not\models \psi \text{ or } \mathfrak{M}, w \models \chi, && \text{by induction hypothesis;}\\
&\Leftrightarrow \mathfrak{M}, w \models \psi \to \chi && \text{by definition of } \mathfrak{M}, w \models \psi \to \chi. \square
\end{aligned}$$

Theorem syn.13. *All tautological instances are valid.* mod:syn:tau:thm:valid-taut

Proof. Contrapositively, suppose φ is such that $\mathfrak{M}, w \not\models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$, for some model $\mathfrak{M}$ and world w. Define an assignment v such that $v(p_i) = 1$ if and only if $\mathfrak{M}, w \models \theta_i$ (and v assigns arbitrary values to $q \notin \{p_1, \dots, p_n\}$). Then by Lemma syn.12, $v \not\models \varphi$, so φ is not a tautology. □

syn.10 Schemas

mod:syn:sch:sec

Definition syn.14. A *schema* is a set of formulas comprising all and only the substitution instances of some modal formula χ, i.e.,

$$\{\psi : \exists\theta_1, \dots, \exists\theta_n\, (\psi = \chi[\theta_1/p_1, \dots, \theta_n/p_n])\}.$$

The formula χ is called the *characteristic* formula of the schema, and it is unique up to a renaming of the propositional variables. A formula φ is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting 'φ', 'ψ', ..., for the atomic components of χ. So, for instance, the following denote schemas: 'φ', '$\varphi \to \Box\varphi$', '$\varphi \to (\psi \to \varphi)$', etc. The schema '$\varphi$' denotes the set of *all* formulas. However, we will also use φ as a meta-linguistic variable for schemas themselves.

Definition syn.15. A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

Theorem syn.16. *The following schema* K *is valid* mod:syn:sch:thm:Kvalid

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi). \tag{K}$$

Valid Schemas	*Invalid Schemas*
$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	$\Box(\varphi \lor \psi) \to (\Box\varphi \lor \Box\psi)$
$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$	$(\Diamond\varphi \land \Diamond\psi) \to \Diamond(\varphi \land \psi)$
$\Box(\varphi \land \psi) \leftrightarrow (\Box\varphi \land \Box\psi)$	$\varphi \to \Box\varphi$
$\Box\varphi \to \Box(\psi \to \varphi)$	$\Box\Diamond\varphi \to \psi$
$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	$\Box\Box\varphi \to \Box\varphi$
$\Diamond(\varphi \lor \psi) \leftrightarrow (\Diamond\varphi \lor \Diamond\psi)$	$\Box\Diamond\varphi \to \Diamond\Box\varphi.$

Figure syn.2: Valid and (or?) invalid schemas.

mod:syn:sch: fig:valid-invalidSchemas

Proof. We need to show that all instances of the schema are true at every world in every model. So let $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that consequent is true as well. In this case, let $\mathfrak{M}, w \models \Box(\varphi \to \psi)$ and $\mathfrak{M}, w \models \Box\varphi$. We need to show $\mathfrak{M} \models \Box\psi$. So let w' be arbitrary such that Rww'. Then by the first assumption $\mathfrak{M}, w' \models \varphi \to \psi$ and by the second assumption $\mathfrak{M}, w' \models \varphi$. It follows that $\mathfrak{M}, w' \models \psi$. Since w' was arbitrary, $\mathfrak{M}, w \models \Box\psi$. □

mod:syn:sch: prop:soundMP

Proposition syn.17. *Show that if φ and $\varphi \to \psi$ are true at a world in a model then so is ψ. Hence, the valid formulas are closed under modus ponens.*

Problem syn.8. Show that none of the following schemas are valid:

D: $\Box\varphi \to \Diamond\varphi$;

T: $\Box\varphi \to \varphi$;

B: $\varphi \to \Box\Diamond\varphi$;

4: $\Box\varphi \to \Box\Box\varphi$;

5: $\Diamond\varphi \to \Box\Diamond\varphi$.

Problem syn.9. Prove that the schemas in the first column of Figure syn.2 are valid and those in the second column are not valid.

Problem syn.10. Decide whether the following schemas are valid or invalid:

1. $(\Diamond\varphi \to \Box\psi) \to (\Box\varphi \to \Box\psi)$;
2. $\Diamond(\varphi \to \psi) \lor \Box(\psi \to \varphi)$.

Problem syn.11. For each of the following schemas find a model $\mathfrak{M}$ such that every instance of the schema is true in $\mathfrak{M}$:

1. $\varphi \to \Diamond\Diamond\varphi$;
2. $\Diamond\varphi \to \Box\varphi$.

syn.11 Frames

mod:syn:fra:sec

Definition syn.18. A *frame* is a pair $\mathfrak{F} = \langle W, R \rangle$ where W is a non-empty set of worlds and R a binary relation on W. A model $\mathfrak{M}$ is *based on* a frame $\mathfrak{F} = \langle W, R \rangle$ if and only if $\mathfrak{M} = \langle W, R, V \rangle$.

Definition syn.19. If $\mathcal{F}$ is a class of frames, we write $\mathcal{F} \vDash \varphi$, "$\varphi$ is valid in $\mathcal{F}$," to mean that φ is true in every model $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathcal{F}$.

The reason frames are interesting is that correspondence between schemas and properties of the accessibility relation R is at the level of frames, *not of models*.

Remark 1. Obviously, if a formula or a schema is valid, i.e., valid with respect to the class of *all* models, it is also valid with respect to any class class $\mathcal{F}$ of frames.

syn.12 Properties of Accessibility Relations

mod:syn:acc:sec

Definition syn.20. We single out the following five potential properties of an accessibility relation:

R is called ...	*... if it satisfies:*
"serial"	$\forall u \exists v\, Ruv$;
"reflexive"	$\forall w\, Rww$;
"symmetric"	$\forall u \forall v (Ruv \rightarrow Rvu)$;
"transitive"	$\forall u \forall v \forall w (Ruv \land Rvw \rightarrow Ruw)$;
"euclidean"	$\forall w \forall u \forall v (Rwu \land Rwv \rightarrow Ruv)$.

Theorem syn.21. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. Then:* *mod:syn:acc:thm:soundschemas*

1. *If R is serial then schema* D*, i.e., $\Box\varphi \rightarrow \Diamond\varphi$, is true in $\mathfrak{M}$;*

2. *If R is reflexive then schema* T*, i.e., $\Box\varphi \rightarrow \varphi$, is true in $\mathfrak{M}$;*

3. *If R is symmetric then schema* B*, i.e., $\varphi \rightarrow \Box\Diamond\varphi$, is true in $\mathfrak{M}$;*

4. *If R is transitive then schema* 4*, i.e., $\Box\varphi \rightarrow \Box\Box\varphi$, is true in $\mathfrak{M}$;*

5. *If R is euclidean then schema* 5*, i.e., $\Diamond\varphi \rightarrow \Box\Diamond\varphi$, is true in $\mathfrak{M}$.*

Proof. Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true all worlds in the model. So let $\varphi \rightarrow \Box\Diamond\varphi$ be a given instance of B, and let $w \in W$ be an arbitrary world. Suppose the antecedent φ is true at w, in order to show that $\Box\Diamond\varphi$ is true at w. So we need to show that $\Diamond\varphi$ is true at all w' accessible from w. Now, for

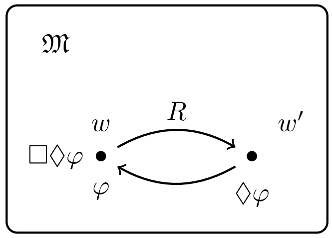


Figure syn.3: The argument from symmetry.

mod:syn:acc: fig:Bsymm

any w' such that Rww' we have, using the hypothesis of symmetry, that also $Rw'w$ (see Figure syn.3). Since $\mathfrak{M}, w \models \varphi$, we have $\mathfrak{M}, w' \models \Diamond\varphi$. Since w' was an arbitrary world such that Rww', we have $\mathfrak{M}, w \models \Box\Diamond\varphi$. □

Problem syn.12. Complete the proof of Theorem syn.21

Notice that the converse implications of Theorem syn.21 do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property (a counterexample is provided by Example syn.22).

mod:syn:acc: ex:reflexive

Example syn.22. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model such that $W = \{u, v\}$, where worlds u and v are related by R: i.e., both Ruv and Rvu. Suppose that for all p: $u \in V(p) \Leftrightarrow v \in V(p)$. Then:

1. For all φ: $\mathfrak{M}, u \models \varphi$ if and only if $\mathfrak{M}, v \models \varphi$ (use induction on φ).

2. Schema T is true in $\mathfrak{M}$.

Since $\mathfrak{M}$ is not reflexive (it is, in fact, *irreflexive*), the converse of Theorem syn.21 fails in the case of T (similar arguments can be given for some—though not all—the other schemas mentioned in Theorem syn.21).

Problem syn.13. Prove the claims in Example syn.22.

syn.13 Frame Correspondence

mod:syn:cor: sec

Even though the converse implications of Theorem syn.21 fail, they hold if we replace "model" by "frame": for the properties considered in Theorem syn.21, it *is* true that if a schema is valid in a *frame* then the accessibility relation of that frame has the corresponding property. In fact, even more is true in the case of D.

mod:syn:cor: ex:D.complete.for.models

Example syn.23. Any model where schema D is true is serial.

Problem syn.14. Prove Example syn.23 (Hint: take $\varphi = \neg\bot$).

If R is ...	*... the following schema is true in* $\mathfrak{M}$*:*
partially functional: $\forall w \forall u \forall v (Rwu \land Rwv \Rightarrow u = v)$	$\Diamond\varphi \to \Box\varphi$;
functional: $\forall u \exists v Ruv$	$\Diamond\varphi \leftrightarrow \Box\varphi$;
weakly dense: $\forall u \forall v (Ruv \Rightarrow \exists w (Ruw \land Rwv))$	$\Box\Box\varphi \to \Box\varphi$;
weakly connected: $\forall w \forall u \forall v ((Rwu \land Rwv) \Rightarrow (Ruv \lor u = v \lor Rvu))$	L: $\Box((\varphi \land \Box\varphi) \to \psi) \lor \Box((\psi \land \Box\psi) \to \varphi)$;
weakly directed: $\forall w \forall u \forall v ((Rwu \land Rwv) \Rightarrow \exists t (Rut \land Rvt)$	G: $\Diamond\Box\varphi \to \Box\Diamond\varphi$;

Figure syn.4: Five more correspondence facts.

mod:syn:cor: fig:anotherfive

Although we will focus on the five classical schemas D, T, B, 4, and 5, we record in Figure syn.4 a few more correspondences.

Problem syn.15. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. Show that if R satisfies the left-hand properties of Figure syn.4, the corresponding right-hand schemas are true in $\mathfrak{M}$.

We now proceed to establish the full correspondence results for frames. We will consider T, B, 4 and 5, as the case for D already follows from Example syn.23.

Theorem syn.24. *Recall that a schema* S *is valid in a frame if each of its instances is true in every model based on that frame. Then:* mod:syn:cor: thm:fullCorrespondence

1. *If* T *is valid in a frame* $\mathfrak{F}$*, then* $\mathfrak{F}$ *is reflexive.*
2. *If* B *is valid in a frame* $\mathfrak{F}$*, then* $\mathfrak{F}$ *is symmetric.*
3. *If* 4 *is valid in a frame* $\mathfrak{F}$*, then* $\mathfrak{F}$ *is transitive.*
4. *If* 5 *is valid in a frame* $\mathfrak{F}$*, then* $\mathfrak{F}$ *is euclidean.*

Proof. The strategy is to devise, for each frame $\mathfrak{F}$, a valuation that will ensure that the frame has the desired property (provided the corresponding schema is true).

1. Suppose T is valid in $\mathfrak{F} = \langle W, R \rangle$, let $w \in W$ be an arbitrary world; we need to show Rww. Fix a propositional variable p and let $u \in V(p)$ if and only if Rwu (when q is other than p, $V(q)$ is arbitrary, say $V(q) = \emptyset$). Let $\mathfrak{M} = \langle W, R, V \rangle$. By construction, for all u such that Rwu: $\mathfrak{M}, u \models p$, and hence $\mathfrak{M}, w \models \Box p$. But by hypothesis $\Box p \to p$, an instance of T, is true at w, so that $\mathfrak{M}, w \models p$, but by definition of V this is possible only if Rww.

2. Suppose B is valid in $\mathfrak{F} = \langle W, R \rangle$, and let $u, v \in W$ be arbitrary worlds such that Ruv; we need to show that Rvu. Fix a propositional variable p, define V such that $w \in V(p)$ if and only if Rvw (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V \rangle$. Notice that the following instance of B: $\neg p \to \Box\Diamond\neg p$, is equivalent to $\Diamond\Box p \to p$. Now, by definition of V, $\mathfrak{M}, w \models p$ for all w such that Rvw, and hence $\mathfrak{M}, v \models \Box p$. Since Ruv, also $\mathfrak{M}, u \models \Diamond\Box p$, and since B is valid in $\mathfrak{F}$, also $\mathfrak{M}, u \models \Diamond\Box p \to p$. It follows that $\mathfrak{M}, u \models p$, whence Rvu, as required.

3. Suppose 4 is valid in $\mathfrak{F} = \langle W, R \rangle$, and let $u, v, w \in W$ be arbitrary worlds such that Ruv and Rvw; we need to show that Ruw. Fix a propositional variable p, define V such that $z \in V(p)$ if and only if Ruz (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V \rangle$. By definition of V, $\mathfrak{M}, z \models p$ for all z such that Ruz, and hence $\mathfrak{M}, u \models \Box p$. But by hypothesis $\Box p \to \Box\Box p$, an instance of 4, is true at u, so that $\mathfrak{M}, u \models \Box\Box p$. Since Ruv and Rvw, we have $\mathfrak{M}, w \models p$, but by definition of V this is possible only if Ruw, as desired.

4. We proceed contrapositively, assuming that the frame $\mathfrak{F} = \langle W, R \rangle$ is not euclidean, and falsifying an instance of 5. Suppose there are worlds u, v, w such that Rwu and Rwv but not Ruv. Fix a propositional variable p and define V such that for all worlds z, $z \in V(p)$ if and only if it is *not* the case that Ruz. Let $\mathfrak{M} = \langle W, R, V \rangle$. Then by hypothesis $\mathfrak{M}, v \models p$ and since Rwv also $\mathfrak{M}, w \models \Diamond p$. However, there is no world y such that Ruy and $\mathfrak{M}, y \models p$ so $\mathfrak{M}, u \models \neg\Diamond p$. Since Rwu, it follows that $\mathfrak{M}, w \not\models \Box\Diamond p$, so that the instance of 5, $\Diamond p \to \Box\Diamond p$ fails at w.

□

Theorem syn.24 also shows that the properties can be combined: for instance if both B and 4 are valid in $\mathfrak{F}$ then the frame is both symmetric and transitive, etc. This is useful because the classical systems **S4** and **S5** are, in fact, just the systems characterized as **KT4** and **KTB4**.

We now record some properties of accessibility relations (in fact, these notions apply to arbitrary binary relations).

mod:syn:cor: prop:relation-facts

Proposition syn.25. *Let R be a binary relation on a set W; then:*

1. *If R is reflexive, then it is serial.*

2. *If R is symmetric, then it is transitive if and only if it is euclidean.*

3. *If R is symmetric or euclidean then it is weakly directed (it has the "diamond property").*

4. *If R is euclidean then it is weakly connected.*

5. *If R is functional then it is serial.*

Problem syn.16. Prove Proposition syn.25.

syn.14 Equivalence Relations and S5

mod:syn:es5:
sec

Definition syn.26. A binary relation R on W is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation R on W is *universal* if and only if Ruv for all $u, v \in W$.

Proposition syn.27. *The following are equivalent:* *mod:syn:es5: prop:equivalences*

1. *R is an equivalence relation;*
2. *R is reflexive and euclidean;*
3. *R is serial, symmetric, and transitive;*
4. *R is serial, symmetric, and euclidean.*

Problem syn.17. Prove Proposition syn.27

Proposition syn.27 is the semantic counterpart to **??**, in that it gives equivalent characterization of the modal logic of frames over which R is an equivalence (the logic traditionally referred to as **S5**).

Proposition syn.28. *Let R be an equivalence relation, and for each $w \in W$ define the* equivalence class *of w as the set $[w] = \{w' \in W : Rww'\}$. Then:*

1. *$w \in [w]$;*
2. *R is universal on each equivalence class $[w]$;*
3. *The collection of equivalence classes partitions W into mutually exclusive and jointly exhaustive subsets.*

Proposition syn.29. *A formula φ is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is an equivalence relation, if and only if it is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is universal. Hence, the logic of universal frames is just* **S5**. *mod:syn:es5: prop:S5=univ*

Proof. It's immediate to verify that a universal relation R on W is an equivalence. Hence, if φ is valid in all frames where R is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose ψ is a formula that fails at a world w in a model $\mathfrak{M} = \langle W, R, V\rangle$ based on a frame $\langle W, R\rangle$, where R is an equivalence on W. So $\mathfrak{M}, w \not\models \psi$. Define a model $\mathfrak{M}' = \langle W', R', V'\rangle$ as follows:

1. $W' = [w]$;
2. R' is universal on W';
3. $V'(p) = V(p) \cap W'$.

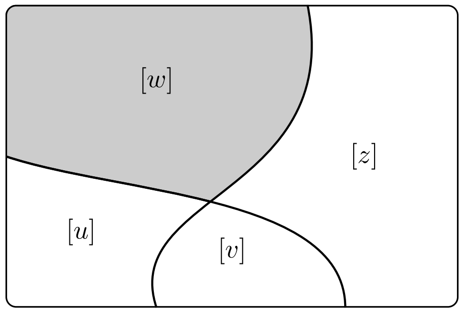


Figure syn.5: A partition of W in equivalence classes.

mod:syn:es5:
fig:partition

(So the set W' of worlds in $\mathfrak{M}'$ is represented by the shaded area in Figure syn.5.) It is easy to see that R and R' agree on W'. Then one can show by induction on formulas that for all $w' \in W'$: $\mathfrak{M}', w' \models \varphi$ if and only if $\mathfrak{M}, w' \models \varphi$ for each φ (this makes sense since $W' \subseteq W$). In particular, $\mathfrak{M}', w \nvDash \psi$, and ψ fails in a model based on a universal frame. $\Box$

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Bibliography