

Chapter udf

Syntax and Semantics of Normal Modal Logics

syn.1 Introduction

mod:syn:int:
sec Modal Logic deals with *modal propositions* and the entailment relations among them. Examples of modal propositions are the following:

1. It is necessary that $2 + 2 = 4$.
2. It is necessarily possible that it will rain tomorrow.
3. If it is necessarily possible that φ then it is possible that φ .

Possibility and necessity are not the only modalities: other unary connectives are also classified as modalities, for instance, “it ought to be the case that φ ,” “It will be the case that φ ,” “Dana knows that φ ,” or “Dana believes that φ .”

Modal logic makes its first appearance in Aristotle’s *De Interpretatione*: he was the first to notice that necessity implies possibility, but not vice versa; that possibility and necessity are inter-definable; that If $\varphi \wedge \psi$ is possibly true then φ is possibly true and ψ is possibly true, but not conversely; and that if $\varphi \rightarrow \psi$ is necessary, then if φ is necessary, so is ψ .

The first modern approach to modal logic was the work of C. I. Lewis, culminating with Lewis and Langford, *Symbolic Logic* (1932). Lewis & Langford were unhappy with the representation of implication by means of the material conditional: $\varphi \rightarrow \psi$ is a poor substitute for “ φ implies ψ .” Instead, they proposed to characterize implication as “Necessarily, if φ then ψ ,” symbolized as $\varphi \rightarrow \psi$. In trying to sort out the different properties, Lewis indentified five different modal systems, **S1**, . . . , **S4**, **S5**, the last two of which are still in use.

The approach of Lewis and Langford was purely *syntactical*: they identified reasonable axioms and rules and investigated what was provable with those means. A semantic approach remained elusive for a long time, until a first attempt was made by Rudolf Carnap in *Meaning and Necessity* (1947) using the notion of a *state description*, i.e., a collection of atomic sentences (those

that are “true” in that state description). After lifting the truth definition to arbitrary sentences φ , Carnap defines φ to be *necessarily true* if it is true in all state descriptions. Carnap’s approach could not handle *iterated* modalities, in that sentences of the form “Possibly necessarily . . . possibly φ ” always reduce to the innermost modality.

The major breakthrough in modal semantics came with Saul Kripke’s article “A Completeness Theorem in Modal Logic” (JSL 1959). Kripke based his work on Leibniz’s idea that a statement is necessarily true if it is true “at all possible worlds.” This idea, though, suffers from the same drawbacks as Carnap’s, in that the truth of statement at a world w (or a state description s) does not depend on w at all. So Kripke assumed that worlds are related by an *accessibility relation* R , and that a statement of the form “Necessarily φ ” is true at a world w if and only if φ is true at all worlds w' *accessible from* w . Semantics that provide some version of this approach are called Kripke semantics and made possible the tumultuous development of modal logics (in the plural).

When interpreted by the Kripke semantics, modal logic shows us what *relational structures* look like “from the inside.” A relational structure is just a set equipped with a binary relation (for instance, the set of students in the class ordered by their social security number is a relational structure). But in fact relational structures come in all sorts of domains: besides relative possibility of states of the world, we can have epistemic states of some agent related by epistemic possibility, or states of a dynamical system with their state transitions, etc. Modal logic can be used to model all of these: the first give us ordinary, alethic, modal logic; the others give us epistemic logic, dynamic logic, etc.

We focus on one particular angle, known to modal logicians as “correspondence theory.” One of the most significant early discoveries of Kripke’s is that many properties of the accessibility relation R (whether it is transitive, symmetric, etc.) can be characterized *in the modal language* itself by means of appropriate “modal schemas.” Modal logicians say, for instance, that the reflexivity of R “corresponds” to the schema “If necessarily φ , then φ ”. We explore mainly the correspondence theory of a number of classical systems of modal logic (e.g., **S4** and **S5**) obtained by a combination of the schemas D, T, B, 4, and 5.

syn.2 The Language of Basic Modal Logic

Definition syn.1. The basic language of modal logic contains

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1. The propositional constant for **falsity** \perp .
2. The propositional constant for **truth** \top .
3. A **denumerable** set of **propositional variables**: p_0, p_1, p_2, \dots

4. The propositional connectives: \neg (negation), \wedge (conjunction), \vee (disjunction), \rightarrow (conditional), \leftrightarrow (biconditional).
5. The modal operator \Box .
6. The modal operator \Diamond .

Definition syn.2. *Formulas* of the basic modal language are inductively defined as follows:

1. \perp is an atomic formula.
2. \top is an atomic formula.
3. Every propositional variable p_i is an (atomic) formula.
4. If φ and ψ are formulas, then $(\varphi \wedge \psi)$ is a formula.
5. If φ and ψ are formulas, then $(\varphi \vee \psi)$ is a formula.
6. If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.
7. If φ and ψ are formulas, then $(\varphi \leftrightarrow \psi)$ is a formula.
8. If φ is a formula, so is $\Box\varphi$.
9. If φ is a formula, then $\Diamond\varphi$ is a formula.
10. Nothing else is a formula.

If a formula φ does not contain \Box or \Diamond , we say it is *modal-free*.

syn.3 Simultaneous Substitution

mod:syn:sub:sec An *instance* of a formula φ is the result of replacing all occurrences of a propositional variable in φ by some other formula. We will refer to instances of formulas often, both when discussing validity and when discussing derivability. It therefore is useful to define the notion precisely.

mod:syn:sub:def:subst-inst **Definition syn.3.** Where φ is a modal formula all of whose propositional variables are among p_1, \dots, p_n , and $\theta_1, \dots, \theta_n$ are also modal formulas, we define $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ as the result of simultaneously substituting each θ_i for p_i in φ . Formally, this is a definition by induction on φ :

1. $\varphi \equiv \perp$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is \perp .
2. $\varphi \equiv \top$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is \top .
3. $\varphi \equiv q$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is q , provided $q \neq p_i$ for $i = 1, \dots, n$.
4. $\varphi \equiv p_i$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is θ_i .

5. $\varphi \equiv \neg\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\neg\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
6. $\varphi \equiv (\psi \wedge \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
 $(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \wedge \chi[\theta_1/p_1, \dots, \theta_n/p_n])$.
7. $\varphi \equiv (\psi \vee \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
 $(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \vee \chi[\theta_1/p_1, \dots, \theta_n/p_n])$.
8. $\varphi \equiv (\psi \rightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
 $(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \rightarrow \chi[\theta_1/p_1, \dots, \theta_n/p_n])$.
9. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
 $(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \leftrightarrow \chi[\theta_1/p_1, \dots, \theta_n/p_n])$.
10. $\varphi \equiv \Box\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Box\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
11. $\varphi \equiv \Diamond\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Diamond\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.

The **formula** $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is called a *substitution instance* of φ .

Example syn.4. Suppose φ is $p_1 \rightarrow \Box(p_1 \wedge p_2)$, θ_1 is $\Diamond(p_2 \rightarrow p_3)$ and θ_2 is $\neg\Box p_1$. Then $\varphi[\theta_1/p_1, \theta_2/p_2]$ is

$$\Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge \neg\Box p_1)$$

while $\varphi[\theta_2/p_1, \theta_1/p_2]$ is

$$\neg\Box p_1 \rightarrow \Box(\neg\Box p_1 \wedge \Diamond(p_2 \rightarrow p_3))$$

Note that simultaneous substitution is in general not the same as iterated substitution, e.g., compare $\varphi[\theta_1/p_1, \theta_2/p_2]$ above with $(\varphi[\theta_1/p_1])[\theta_2/p_2]$, which is:

$$\begin{aligned} &\Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge p_2)[\neg\Box p_1/p_2], \text{ i.e.,} \\ &\Diamond(\neg\Box p_1 \rightarrow p_3) \rightarrow \Box(\Diamond(\neg\Box p_1 \rightarrow p_3) \wedge \neg\Box p_1) \end{aligned}$$

and with $(\varphi[\theta_2/p_2])[\theta_1/p_1]$:

$$\begin{aligned} &p_1 \rightarrow \Box(p_1 \wedge \neg\Box p_1)[\Diamond(p_2 \rightarrow p_3)/p_1], \text{ i.e.,} \\ &\Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge \neg\Box\Diamond(p_2 \rightarrow p_3)). \end{aligned}$$

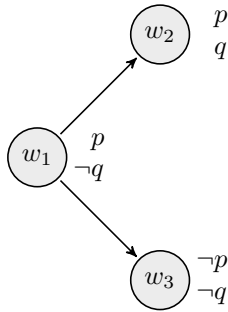


Figure syn.1: A simple model.

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fig:simple

syn.4 Relational Models

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The basic concept of semantics for normal modal logics is that of a *relational model*. It consists of a set of worlds, which are related by a binary “accessibility relation,” together with an assignment which determines which **propositional variables** count as “true” at which worlds.

Definition syn.5. A *model* for the basic modal language is a triple $\mathfrak{M} = \langle W, R, V \rangle$, where

1. W is a nonempty set of “worlds,”
2. R is a binary accessibility relation on W , and
3. V is a function assigning to each **propositional variable** p a set $V(p)$ of possible worlds.

When Rww' holds, we say that w' is *accessible from* w . When $w \in V(p)$ we say p is *true at* w .

The great advantage of relational semantics is that models can be represented by means of simple diagrams, such as the one in [Figure syn.1](#). Worlds are represented by nodes, and world w' is accessible from w precisely when there is an arrow from w to w' . Moreover, we label a node (world) by p when $w \in V(p)$, and otherwise by $\neg p$. [Figure syn.1](#) represents the model with $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2 \rangle, \langle w_1, w_3 \rangle\}$, $V(p) = \{w_1, w_2\}$, and $V(q) = \{w_2\}$.

syn.5 Truth at a World

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Every modal model determines which modal **formulas** count as true at which worlds in it. The relation “model \mathfrak{M} makes **formula** φ true at world w ” is the basic notion of relational semantics. The relation is defined inductively and

coincides with the usual characterization using truth tables for the non-modal operators.

Definition syn.6. *Truth of a formula* φ at w in a \mathfrak{M} , in symbols: $\mathfrak{M}, w \Vdash \varphi$, is defined inductively as follows: [mod:syn:trw:](#)
[defn:mmodels](#)

1. $\varphi \equiv \perp$: Never $\mathfrak{M}, w \Vdash \perp$.
2. $\varphi \equiv \top$: Always $\mathfrak{M}, w \Vdash \top$.
3. $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \not\Vdash \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \not\Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff either both $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$ or neither $\mathfrak{M}, w \Vdash \psi$ nor $\mathfrak{M}, w \Vdash \chi$.
9. $\varphi \equiv \Box\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for all $w' \in W$ with Rww' [mod:syn:trw:](#)
[defn:sub:mmodels-box](#)
10. $\varphi \equiv \Diamond\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for at least one $w' \in W$ with Rww' [mod:syn:trw:](#)
[defn:sub:mmodels-diamond](#)

Note that by clause (9), a formula $\Box\psi$ is true at w whenever there are no w' with wRw' . In such a case $\Box\psi$ is *vacuously* true at w . Also, $\Box\psi$ may be satisfied at w even if ψ is not. The truth of ψ at w does not guarantee the truth of $\Diamond\psi$ at w . This holds, however, if Rww , e.g., if R is reflexive. If there is no w' such that Rww' , then $\mathfrak{M}, w \not\Vdash \Diamond\varphi$, for any φ .

Problem syn.1. Consider the model of [Figure syn.1](#). Which of the following hold?

1. $\mathfrak{M}, w_1 \Vdash q$;
2. $\mathfrak{M}, w_3 \Vdash \neg q$;
3. $\mathfrak{M}, w_1 \Vdash p \vee q$;
4. $\mathfrak{M}, w_1 \Vdash \Box(p \vee q)$;
5. $\mathfrak{M}, w_3 \Vdash \Box q$;
6. $\mathfrak{M}, w_3 \Vdash \Box \perp$;
7. $\mathfrak{M}, w_1 \Vdash \Diamond q$;
8. $\mathfrak{M}, w_1 \Vdash \Box q$;
9. $\mathfrak{M}, w_1 \Vdash \neg \Box \Box \neg q$.

mod:syn:tru:
prop:dual

Proposition syn.7.

1. $\mathfrak{M}, w \Vdash \Box\varphi$ iff $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$.
2. $\mathfrak{M}, w \Vdash \Diamond\varphi$ iff $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$.

Proof. 1. $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$ iff $\mathfrak{M} \not\Vdash \Diamond\neg\varphi$ by definition of $\mathfrak{M}, w \Vdash$. $\mathfrak{M}, w \Vdash \Diamond\neg\varphi$ iff for some w' with Rww' , $\mathfrak{M}, w' \Vdash \neg\varphi$. Hence, $\mathfrak{M}, w \not\Vdash \Diamond\neg\varphi$ iff for all w' with Rww' , $\mathfrak{M}, w' \not\Vdash \neg\varphi$. We also have $\mathfrak{M}, w' \not\Vdash \neg\varphi$ iff $\mathfrak{M}, w' \Vdash \varphi$. Together we have $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$ iff for all w' with Rww' , $\mathfrak{M}, w' \Vdash \varphi$. Again by definition of $\mathfrak{M}, w \Vdash$, that is the case iff $\mathfrak{M}, w \Vdash \Box\varphi$.

2. $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$ iff $\mathfrak{M} \not\Vdash \Box\neg\varphi$. $\mathfrak{M}, w \Vdash \Box\neg\varphi$ iff for all w' with Rww' , $\mathfrak{M}, w' \Vdash \neg\varphi$. Hence, $\mathfrak{M}, w \not\Vdash \Box\neg\varphi$ iff for some w' with Rww' , $\mathfrak{M}, w' \not\Vdash \neg\varphi$. We also have $\mathfrak{M}, w' \not\Vdash \neg\varphi$ iff $\mathfrak{M}, w' \Vdash \varphi$. Together we have $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$ iff for some w' with Rww' , $\mathfrak{M}, w' \Vdash \varphi$. Again by definition of $\mathfrak{M}, w \Vdash$, that is the case iff $\mathfrak{M}, w \Vdash \Diamond\varphi$. □

Problem syn.2. Complete the proof of Proposition syn.7.

Problem syn.3. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model, and suppose $w_1, w_2 \in W$ are such that:

1. $w_1 \in V(p)$ if and only if $w_2 \in V(p)$; and
2. for all $w \in W$: Rw_1w if and only if Rw_2w .

Using induction on formulas, show that for all formulas φ : $\mathfrak{M}, w_1 \Vdash \varphi$ if and only if $\mathfrak{M}, w_2 \Vdash \varphi$.

Problem syn.4. Let $\mathfrak{M} = \langle M, R, V \rangle$. Show that $\mathfrak{M}, w \Vdash \neg\Diamond\varphi$ if and only if $\mathfrak{M}, w \Vdash \Box\neg\varphi$.

syn.6 Truth in a Model

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Sometimes we are interested which formulas are true at every world in a given model. Let's introduce a notation for this.

Definition syn.8. A formula φ is true in a model $M = \langle W, R, V \rangle$, written $\mathfrak{M} \Vdash \varphi$, if and only if $\mathfrak{M}, w \Vdash \varphi$ for every $w \in W$.

mod:syn:tru:
prop:truthfacts

Proposition syn.9.

1. If $\mathfrak{M} \Vdash \varphi$ then $\mathfrak{M} \not\Vdash \neg\varphi$, but not vice-versa.
2. If $\mathfrak{M} \Vdash \varphi \rightarrow \psi$ then $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but not vice-versa.

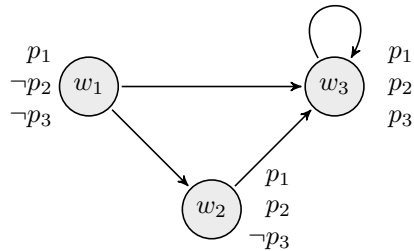
Proof. 1. If $\mathfrak{M} \Vdash \varphi$ then φ is true at all worlds in W , and since $W \neq \emptyset$, it can't be that $\mathfrak{M} \Vdash \neg\varphi$, or else φ would have to be both true and false at some world.

On the other hand, if $\mathfrak{M} \not\Vdash \neg\varphi$ then φ is true at some world $w \in W$. It does not follow that $\mathfrak{M}, w \Vdash \varphi$ for every $w \in W$. For instance, in the model of Figure syn.1, $\mathfrak{M} \not\Vdash \neg p$, and also $\mathfrak{M} \not\Vdash p$.

2. Assume $\mathfrak{M} \Vdash \varphi \rightarrow \psi$ and $\mathfrak{M} \Vdash \varphi$; to show $\mathfrak{M} \Vdash \psi$ let $w \in W$ be an arbitrary world. Then $\mathfrak{M}, w \Vdash \varphi \rightarrow \psi$ and $\mathfrak{M}, w \Vdash \varphi$, so $\mathfrak{M}, w \Vdash \psi$, and since w was arbitrary, $\mathfrak{M} \Vdash \psi$.

To show that the converse fails, we need to find a model \mathfrak{M} such that $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but $\mathfrak{M} \not\Vdash \varphi \rightarrow \psi$. Consider again the model of Figure syn.1: $\mathfrak{M} \not\Vdash p$ and hence (vacuously) $\mathfrak{M} \Vdash p$ only if $\mathfrak{M} \Vdash q$. However, $\mathfrak{M} \not\Vdash p \rightarrow q$, as p is true but q false at w_1 . □

Problem syn.5. Consider the following model \mathfrak{M} for the language comprising p_1, p_2, p_3 as the only propositional variables:



Are the following formulas and schemas true in the model \mathfrak{M} , i.e., true at every world in \mathfrak{M} ? Explain.

1. $p \rightarrow \Diamond p$ (for p atomic);
2. $\varphi \rightarrow \Diamond \varphi$ (for φ arbitrary);
3. $\Box p \rightarrow p$ (for p atomic);
4. $\neg p \rightarrow \Diamond \Box p$ (for p atomic);
5. $\Diamond \Box \varphi$ (for φ arbitrary);
6. $\Box \Diamond p$ (for p atomic).

syn.7 Validity

mod:syn:val: sec **Formulas** that are true in all models, i.e., true at every world in every model, explanation are particularly interesting. They represent those modal propositions which are true regardless of how \Box and \Diamond are interpreted, as long as the interpretation is “normal” in the sense that it is generated by some accessibility relation on possible worlds. We call such **formulas** *valid*. For instance, $\Box(p \wedge q) \rightarrow \Box p$ is valid. Some **formulas** one might expect to be valid on the basis of the alethic interpretation of \Box , such as $\Box p \rightarrow p$, are not valid, however. Part of the interest of relational models is that different interpretations of \Box and \Diamond can be captured by different kinds of accessibility relations. This suggests that we should define validity not just relative to *all* models, but relative to all models of *a certain kind*. It will turn out, e.g., that $\Box p \rightarrow p$ is true in all models where every world is accessible from itself, i.e., R is reflexive. Defining validity relative to classes of models enables us to formulate this succinctly: $\Box p \rightarrow p$ is valid in the class of reflexive models.

Definition syn.10. A formula φ is *valid* in a class \mathcal{C} of models if it is true in every model in \mathcal{C} (i.e., true at every world in every model in \mathcal{C}). If φ is valid in \mathcal{C} , we write $\mathcal{C} \models \varphi$, and we write $\models \varphi$ if φ is valid in the class of *all* models.

Proposition syn.11. If φ is valid in \mathcal{C} it is also valid in each class $\mathcal{C}' \subseteq \mathcal{C}$.

mod:syn:val: prop:Nec-rule **Proposition syn.12.** If φ is valid, then so is $\Box\varphi$.

Proof. Assume $\models \varphi$. To show $\models \Box\varphi$ let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and $w \in W$. If Rww' then $\mathfrak{M}, w' \models \varphi$, since φ is valid, and so also $\mathfrak{M}, w \models \Box\varphi$. Since \mathfrak{M} and w were arbitrary, $\models \Box\varphi$. \square

Problem syn.6. Show that the following are valid:

1. $\models \Box p \rightarrow \Box(q \rightarrow p)$;
2. $\models \Box \neg \perp$;
3. $\models \Box p \rightarrow (\Box q \rightarrow \Box p)$.

Problem syn.7. Show that $\varphi \rightarrow \Box\varphi$ is valid in the class \mathcal{C} of models $\mathfrak{M} = \langle W, R, V \rangle$ where $W = \{w\}$. Similarly, show that $\psi \rightarrow \Box\psi$ and $\Diamond\psi \rightarrow \psi$ are valid in the class of models $\mathfrak{M} = \langle W, R, V \rangle$ where $R = \emptyset$.

syn.8 Tautological Instances

mod:syn:tau: sec A modal-free formula is a tautology if it is true under every truth-value assignment. Clearly, every tautology is true at every world in every model. But for explanation **formulas** involving \Box and \Diamond , the notion of tautology is not defined. Is it the case, e.g., that $\Box p \vee \neg\Box p$ —an instance of the principle of excluded middle—is valid? The notion of a *tautological instance* helps: a **formula** that is a substitution instance of a (non-modal) tautology. It is not surprising, but still requires proof, that every tautological instance is valid.

Definition syn.13. A modal **formula** ψ is a *tautological instance* if and only if there is a modal-free tautology φ with **propositional variables** p_1, \dots, p_n and **formulas** $\theta_1, \dots, \theta_n$ such that $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.

Lemma syn.14. *Suppose φ is a modal-free **formula** whose propositional variables are p_1, \dots, p_n , and let $\theta_1, \dots, \theta_n$ be modal **formulas**. Then for any assignment \mathbf{v} , any model $\mathfrak{M} = \langle W, R, V \rangle$, and any $w \in W$ such that $\mathbf{v}(p_i) = \top$ if and only if $\mathfrak{M}, w \Vdash \theta_i$ we have that $\mathbf{v} \models \varphi$ if and only if $\mathfrak{M}, w \Vdash \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.* mod:syn:tau:
lem:valid-taut

Proof. By induction on φ .

1. $\varphi \equiv \perp$: Both $\mathbf{v} \not\models \perp$ and $\mathfrak{M}, w \not\Vdash \perp$.

2. $\varphi \equiv \top$: Both $\mathbf{v} \models \top$ and $\mathfrak{M}, w \Vdash \top$.

3. $\varphi \equiv p_i$:

$$\begin{aligned} \mathbf{v} \models p_i &\Leftrightarrow \mathbf{v}(p_i) = \top && \text{by definition of } \mathbf{v} \models p_i; \\ &\Leftrightarrow \mathfrak{M}, w \Vdash \theta_i && \text{by assumption} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash p_i[\theta_1/p_1, \dots, \theta_n/p_n] && \text{since } p_i[\theta_1/p_1, \dots, \theta_n/p_n] \equiv \theta_i. \end{aligned}$$

4. $\varphi \equiv \neg\psi$:

$$\begin{aligned} \mathbf{v} \models \neg\psi &\Leftrightarrow \mathbf{v} \not\models \psi && \text{by definition of } \mathbf{v} \models; \\ &\Leftrightarrow \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by induction hypothesis;} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash \neg\psi[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathbf{v} \models. \end{aligned}$$

5. $\varphi \equiv (\psi \wedge \chi)$:

$$\begin{aligned} \mathbf{v} \models \psi \wedge \chi &\Leftrightarrow \mathbf{v} \models \psi \text{ and } \mathbf{v} \models \chi && \text{by definition of } \mathbf{v} \models; \\ &\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} && \\ &\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n], && \text{by induction hypothesis;} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \wedge \chi)[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{M}, w \Vdash. \end{aligned}$$

6. $\varphi \equiv (\psi \vee \chi)$:

$$\begin{aligned} \mathbf{v} \models \psi \vee \chi &\Leftrightarrow \mathbf{v} \models \psi \text{ or } \mathbf{v} \models \chi && \text{by definition of } \mathbf{v} \models; \\ &\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} && \\ &\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n], && \text{by induction hypothesis;} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \vee \chi)[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{M}, w \Vdash. \end{aligned}$$

7. $\varphi \equiv (\psi \rightarrow \chi)$:

$$\begin{aligned} \mathbf{v} \models \psi \rightarrow \chi &\Leftrightarrow \mathbf{v} \not\models \psi \text{ or } \mathbf{v} \models \chi && \text{by definition of } \mathbf{v} \models; \\ &\Leftrightarrow \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} && \\ &\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n], && \text{by induction hypothesis;} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \rightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{M}, w \Vdash. \end{aligned}$$

8. $\varphi \equiv (\psi \leftrightarrow \chi)$:

$$\begin{aligned}
\mathbf{v} \models \psi \rightarrow \chi &\Leftrightarrow \text{either } \mathbf{v} \models \psi \text{ and } \mathbf{v} \models \chi \\
&\text{or } \mathbf{v} \not\models \psi \text{ and } \mathbf{v} \not\models \chi && \text{by definition of } \mathbf{v} \models; \\
&\Leftrightarrow \text{either } \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{or } \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \not\Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n], && \text{by induction hypothesis;} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \leftrightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] && \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

□

mod:syn:tau: **Proposition syn.15.** *All tautological instances are valid.*
prop:valid-taut

Proof. Contrapositively, suppose φ is such that $\mathfrak{M}, w \not\Vdash \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$, for some model \mathfrak{M} and world w . Define an assignment v such that $v(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \Vdash \theta_i$ (and v assigns arbitrary values to $q \notin \{p_1, \dots, p_n\}$). Then by [Lemma syn.14](#), $\mathbf{v} \not\models \varphi$, so φ is not a tautology. □

syn.9 Schemas and Validity

mod:syn:sch:
sec

Definition syn.16. A *schema* is a set of **formulas** comprising all and only the substitution instances of some modal **formula** χ , i.e.,

$$\{\psi : \exists \theta_1, \dots, \exists \theta_n (\psi = \chi[\theta_1/p_1, \dots, \theta_n/p_n])\}.$$

The **formula** χ is called the *characteristic formula* of the schema, and it is unique up to a renaming of the propositional variables. A **formula** φ is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting ‘ φ ’, ‘ ψ ’, \dots , for the atomic components of χ . So, for instance, the following denote schemas: ‘ φ ’, ‘ $\varphi \rightarrow \Box\varphi$ ’, ‘ $\varphi \rightarrow (\psi \rightarrow \varphi)$ ’. They correspond to the characteristic **formulas** p , $p \rightarrow \Box p$, $p \rightarrow (q \rightarrow p)$. The schema ‘ φ ’ denotes the set of *all formulas*.

Definition syn.17. A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

mod:syn:sch: **Proposition syn.18.** *The following schema K is valid*
prop:Kvalid

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi). \quad (\text{K})$$

Proof. We need to show that all instances of the schema are true at every world in every model. So let $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that consequent is true as well. In this case, let $\mathfrak{M}, w \Vdash \Box(\varphi \rightarrow \psi)$ and $\mathfrak{M}, w \Vdash \Box\varphi$. We need to show $\mathfrak{M} \Vdash \Box\psi$. So let w' be arbitrary such that Rww' . Then by the first assumption $\mathfrak{M}, w' \Vdash \varphi \rightarrow \psi$ and by the second assumption $\mathfrak{M}, w' \Vdash \varphi$. It follows that $\mathfrak{M}, w' \Vdash \psi$. Since w' was arbitrary, $\mathfrak{M}, w \Vdash \Box\psi$. \square

Proposition syn.19. *The following schema DUAL is valid*

*mod:syn:sch:
prop: Dual-valid*

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi. \quad (\text{DUAL})$$

Proof. Exercise. \square

Problem syn.8. Prove Proposition syn.19.

Proposition syn.20. *If φ and $\varphi \rightarrow \psi$ are true at a world in a model then so is ψ . Hence, the valid formulas are closed under modus ponens.*

*mod:syn:sch:
prop:soundMP*

Proposition syn.21. *A formula φ is valid iff all its substitution instances are. In other words, a schema is valid iff its characteristic formula is.*

*mod:syn:sch:
prop:valid-instances*

Proof. The “if” direction is obvious, since φ is a substitution instance of itself.

To prove the “only if” direction, we show the following: Suppose $\mathfrak{M} = \langle W, R, V \rangle$ is a modal model, and $\psi \equiv \varphi[D_1/p_1, \dots, D_n/p_n]$ is a substitution instance of φ . Define $\mathfrak{M}' = \langle W, R, V' \rangle$ by $V(p_i) = \{w : \mathfrak{M} \Vdash \theta_i w\}$. Then $\mathfrak{M} \Vdash \psi w$ iff $\mathfrak{M}' \Vdash \varphi w$, for any $w \in W$. (We leave the proof as an exercise.) Now suppose that φ was valid, but some substitution instance ψ of φ was not valid. Then for some $\mathfrak{M} = \langle W, R, V \rangle$ and some $w \in W$, $\mathfrak{M} \not\Vdash \psi w$. But then $\mathfrak{M}' \not\Vdash \varphi w$ by the claim, and φ is not valid, a contradiction. \square

Problem syn.9. Prove the claim in the “only if” part of the proof of Proposition syn.21. (Hint: use induction on φ .)

Note, however, that it is not true that a schema is true in a model iff its characteristic formula is. Of course, the “only if” direction holds: if every instance of φ is true in \mathfrak{M} , φ itself is true in \mathfrak{M} . But it may happen that φ is true in \mathfrak{M} but some instance of φ is false at some world in \mathfrak{M} . For a very simple counterexample consider p in a model with only one world w and $V(p) = \{w\}$, so that p is true at w . But \perp is an instance of p , and not true at w .

Problem syn.10. Show that none of the following formulas are valid:

- D: $\Box p \rightarrow \Diamond p$;
- T: $\Box p \rightarrow p$;
- B: $p \rightarrow \Box \Diamond p$;
- 4: $\Box p \rightarrow \Box \Box p$;

Valid Schemas	Invalid Schemas
$\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$	$\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Box\psi)$
$\Diamond(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Diamond\psi)$	$(\Diamond\varphi \wedge \Diamond\psi) \rightarrow \Diamond(\varphi \wedge \psi)$
$\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$	$\varphi \rightarrow \Box\varphi$
$\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$	$\Box\Diamond\varphi \rightarrow \psi$
$\neg\Diamond\varphi \rightarrow \Box(\varphi \rightarrow \psi)$	$\Box\Box\varphi \rightarrow \Box\varphi$
$\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$	$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

Table syn.1: Valid and (or?) invalid schemas.

mod:syn:sch:
tab:valid-invalidSchemas

5: $\Diamond p \rightarrow \Box\Diamond p$.

Problem syn.11. Prove that the schemas in the first column of table syn.1 are valid and those in the second column are not valid.

Problem syn.12. Decide whether the following schemas are valid or invalid:

1. $(\Diamond\varphi \rightarrow \Box\psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$;
2. $\Diamond(\varphi \rightarrow \psi) \vee \Box(\psi \rightarrow \varphi)$.

Problem syn.13. For each of the following schemas find a model \mathfrak{M} such that every instance of the formula is true in \mathfrak{M} :

1. $p \rightarrow \Diamond\Diamond p$;
2. $\Diamond p \rightarrow \Box p$.

syn.10 Entailment

mod:syn:ent:
sec With the definition of truth at a world, we can define an entailment relation [explanation](#) between formulas. A formula ψ entails φ iff, whenever ψ is true, φ is true as well. Here, “whenever” means both “whichever model we consider” as well as “whichever world in that model we consider.”

Definition syn.22. If Γ is a set of formulas and φ a formula, then Γ entails φ , in symbols: $\Gamma \vDash \varphi$, if and only if for every model $\mathfrak{M} = \langle W, R, V \rangle$ and world $w \in W$, if $\mathfrak{M}, w \Vdash \psi$ for every $\psi \in \Gamma$, then $\mathfrak{M}, w \Vdash \varphi$. If Γ contains a single formula ψ , then we write $\psi \vDash \varphi$.

Example syn.23. To show that a formula entails another, we have to reason about all models, using the definition of $\mathfrak{M}, w \Vdash$. For instance, to show $p \rightarrow \Diamond p \vDash \Box\neg p \rightarrow \neg p$, we might argue as follows: Consider a model $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$, and suppose $\mathfrak{M}, w \Vdash p \rightarrow \Diamond p$. We have to show that $\mathfrak{M}, w \Vdash \Box\neg p \rightarrow \neg p$. Suppose not. Then $\mathfrak{M}, w \Vdash \Box\neg p$ and $\mathfrak{M}, w \not\Vdash \neg p$. Since $\mathfrak{M}, w \not\Vdash \neg p$, $\mathfrak{M}, w \Vdash p$. By assumption, $\mathfrak{M}, w \Vdash p \rightarrow \Diamond p$, hence $\mathfrak{M}, w \Vdash \Diamond p$. By definition of $\mathfrak{M}, w \Vdash \Diamond p$, there is some w' with Rww' such that $\mathfrak{M}, w' \Vdash p$. Since also $\mathfrak{M}, w \Vdash \Box\neg p$, $\mathfrak{M}, w' \Vdash \neg p$, a contradiction.

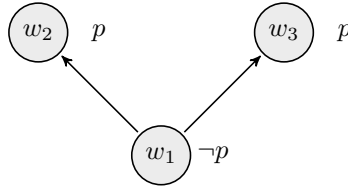


Figure syn.2: Counterexample to $p \rightarrow \Diamond p \vDash \Box p \rightarrow p$.

mod:syn:ent:
fig:counterex

To show that a formula ψ does not entail another φ , we have to give a counterexample, i.e., a model $\mathfrak{M} = \langle W, R, V \rangle$ where we show that at some world $w \in W$, $\mathfrak{M}, w \Vdash \psi$ but $\mathfrak{M}, w \not\Vdash \varphi$. Let's show that $p \rightarrow \Diamond p \not\vDash \Box p \rightarrow p$. Consider the model in Figure syn.2. We have $\mathfrak{M}, w_1 \Vdash \Diamond p$ and hence $\mathfrak{M}, w_1 \Vdash p \rightarrow \Diamond p$. However, since $\mathfrak{M}, w_1 \Vdash \Box p$ but $\mathfrak{M}, w_1 \not\Vdash p$, we have $\mathfrak{M}, w_1 \not\vDash \Box p \rightarrow p$.

Often very simple counterexamples suffice. The model $\mathfrak{M}' = \{W', R', V'\}$ with $W' = \{w\}$, $R' = \emptyset$, and $V'(p) = \emptyset$ is also a counterexample: Since $\mathfrak{M}', w \not\Vdash p$, $\mathfrak{M}', w \Vdash p \rightarrow \Diamond p$. As no worlds are accessible from w , we have $\mathfrak{M}', w \Vdash \Box p$, and so $\mathfrak{M}', w \not\vDash \Box p \rightarrow p$.

Problem syn.14. Show that $\Box(\varphi \wedge \psi) \vDash \Box\varphi$.

Problem syn.15. Show that $\Box(p \rightarrow q) \not\vDash p \rightarrow \Box q$ and $p \rightarrow \Box q \not\vDash \Box(p \rightarrow q)$.

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Bibliography