syn.1 Simultaneous Substitution

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An *instance* of a formula φ is the result of replacing all occurrences of a propositional variable in φ by some other formula. We will refer to instances of formulas often, both when discussing validity and when discussing derivability. It therefore is useful to define the notion precisely.

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Definition syn.1. Where φ is a modal formula all of whose propositional variables are among $p_1, \dots, p_n$, and $\theta_1, \dots, \theta_n$ are also modal formulas, we define $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ as the result of simultaneously substituting each θ_i for p_i in A. Formally, this is a definition by induction on φ:

1. $\varphi \equiv \bot$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\bot$.
2. $\varphi \equiv \top$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\top$.
3. $\varphi \equiv q$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is q, provided $q \not\equiv p_i$ for $i = 1, \dots, n$.
4. $\varphi \equiv p_i$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is θ_i.
5. $\varphi \equiv \neg\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\neg\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
6. $\varphi \equiv (\psi \land \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \land \chi[\theta_1/p_1, \dots, \theta_n/p_n]).$$
7. $\varphi \equiv (\psi \lor \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \lor \chi[\theta_1/p_1, \dots, \theta_n/p_n]).$$
8. $\varphi \equiv (\psi \to \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \to \chi[\theta_1/p_1, \dots, \theta_n/p_n]).$$
9. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \leftrightarrow \chi[\theta_1/p_1, \dots, \theta_n/p_n]).$$
10. $\varphi \equiv \Box\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Box\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
11. $\varphi \equiv \Diamond\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Diamond\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.

The formula $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is called a *substitution instance* of φ.

Example syn.2. Suppose φ is $p_1 \to \Box(p_1 \land p_2)$, θ_1 is $\Diamond(p_2 \to p_3)$ and θ_2 is $\neg\Box p_1$. Then $\varphi[\theta_1/p_1, \theta_2/p_2]$ is

$$\Diamond(p_2 \to p_3) \to \Box(\Diamond(p_2 \to p_3) \land \neg\Box p_1)$$

while $\varphi[\theta_2/p_1, \theta_1/p_2]$ is

$$\neg\Box p_1 \to \Box(\neg\Box p_1 \land \Diamond(p_2 \to p_3))$$

Note that simultaneous substitution is in general not the same as iterated substitution, e.g., compare $\varphi[\theta_1/p_1, \theta_2/p_2]$ above with $(\varphi[\theta_1/p_1])[\theta_2/p_2]$, which is:

$$\begin{aligned} \Diamond(p_2 \to p_3) \to \Box(\Diamond(p_2 \to p_3) \land p_2)[\neg\Box p_1/p_2], \text{ i.e.,} \\ \Diamond(\neg\Box p_1 \to p_3) \to \Box(\Diamond(\neg\Box p_1 \to p_3) \land \neg\Box p_1) \end{aligned}$$

and with $(\varphi[\theta_2/p_2])[\theta_1/p_1]$:

$$\begin{aligned} p_1 \to \Box(p_1 \land \neg\Box p_1)[\Diamond(p_2 \to p_3)/p_1], \text{ i.e.,} \\ \Diamond(p_2 \to p_3) \to \Box(\Diamond(p_2 \to p_3) \land \neg\Box\Diamond(p_2 \to p_3)). \end{aligned}$$

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Bibliography