## syn.1 Simultaneous Substitution

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An *instance* of a formula $\varphi$ is the result of replacing all occurrences of a propositional variable in $\varphi$ by some other formula. We will refer to instances of formulas often, both when discussing validity and when discussing derivability. It therefore is useful to define the notion precisely.

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**Definition syn.1.** Where $\varphi$ is a modal formula all of whose propositional variables are among $p_1, \dots, p_n$, and $\theta_1, \dots, \theta_n$ are also modal formulas, we define $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ as the result of simultaneously substituting each $\theta_i$ for $p_i$ in $A$. Formally, this is a definition by induction on $\varphi$:

1. $\varphi \equiv \bot$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\bot$.
2. $\varphi \equiv \top$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\top$.
3. $\varphi \equiv q$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $q$, provided $q \not\equiv p_i$ for $i = 1, \dots, n$.
4. $\varphi \equiv p_i$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\theta_i$.
5. $\varphi \equiv \neg\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\neg\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
6. $\varphi \equiv (\psi \land \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \land \theta[\theta_1/p_1, \dots, \theta_n/p_n]).$$
7. $\varphi \equiv (\psi \lor \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \lor \theta[\theta_1/p_1, \dots, \theta_n/p_n]).$$
8. $\varphi \equiv (\psi \rightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \rightarrow \theta[\theta_1/p_1, \dots, \theta_n/p_n]).$$
9. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \leftrightarrow \theta[\theta_1/p_1, \dots, \theta_n/p_n]).$$
10. $\varphi \equiv \Box\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Box\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
11. $\varphi \equiv \Diamond\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Diamond\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.

The formula $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is called a *substitution instance* of $\varphi$.

**Example syn.2.** Suppose $\varphi$ is $p_1 \rightarrow \Box(p_1 \land p_2)$, $\theta_1$ is $\Diamond(p_2 \rightarrow p_3)$ and $D_2$ is $\neg\Box p_1$. Then $\varphi[\theta_1/p_1, \theta_2/p_2]$ is

$$\Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \land \neg\Box p_1)$$

while $\varphi[\theta_2/p_1, \theta_1/p_2]$ is

$$\neg\Box p_1 \to \Box(\neg\Box p_1 \land \Diamond(p_2 \to p_3))$$

Note that simultaneous substitution is in general not the same as iterated substitution, e.g., compare $\varphi[\theta_1/p_1, \theta_2/p_2]$ above with $\varphi[\theta_1/p_1][\theta_2/p_2]$:

$$\Diamond(\neg\Box p_1 \to p_3) \to \Box(\Diamond(\neg\Box p_1 \to p_3) \land \neg\Box p_1)$$

and with $\varphi[\theta_2/p_2][\theta_1/p_1]$:

$$\Diamond(\neg\Box p_1 \to p_3) \to \Box(\Diamond(\neg\Box p_1 \to p_3) \land \neg\Box\Diamond(\neg\Box p_1 \to p_3))$$

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## Bibliography