

syn.1 Simultaneous Substitution

nml:syn:sub:sec An *instance* of a **formula** φ is the result of replacing all occurrences of a **propositional variable** in φ by some other **formula**. We will refer to instances of **formulas** often, both when discussing validity and when discussing **derivability**. It therefore is useful to define the notion precisely.

nml:syn:sub: def:subst-inst **Definition syn.1.** Where φ is a modal **formula** all of whose **propositional variables** are among p_1, \dots, p_n , and $\theta_1, \dots, \theta_n$ are also modal **formulas**, we define $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ as the result of simultaneously substituting each θ_i for p_i in φ . Formally, this is a definition by induction on φ :

1. $\varphi \equiv \perp$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is \perp .
2. $\varphi \equiv \top$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is \top .
3. $\varphi \equiv q$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is q , provided $q \neq p_i$ for $i = 1, \dots, n$.
4. $\varphi \equiv p_i$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is θ_i .
5. $\varphi \equiv \neg\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\neg\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
6. $\varphi \equiv (\psi \wedge \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is

$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \wedge \chi[\theta_1/p_1, \dots, \theta_n/p_n]).$$
7. $\varphi \equiv (\psi \vee \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is

$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \vee \chi[\theta_1/p_1, \dots, \theta_n/p_n]).$$
8. $\varphi \equiv (\psi \rightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is

$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \rightarrow \chi[\theta_1/p_1, \dots, \theta_n/p_n]).$$
9. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is

$$(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \leftrightarrow \chi[\theta_1/p_1, \dots, \theta_n/p_n]).$$
10. $\varphi \equiv \Box\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Box\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
11. $\varphi \equiv \Diamond\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Diamond\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.

The **formula** $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is called a *substitution instance* of φ .

Example syn.2. Suppose φ is $p_1 \rightarrow \Box(p_1 \wedge p_2)$, θ_1 is $\Diamond(p_2 \rightarrow p_3)$ and θ_2 is $\neg\Box p_1$. Then $\varphi[\theta_1/p_1, \theta_2/p_2]$ is

$$\Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge \neg\Box p_1)$$

while $\varphi[\theta_2/p_1, \theta_1/p_2]$ is

$$\neg\Box p_1 \rightarrow \Box(\neg\Box p_1 \wedge \Diamond(p_2 \rightarrow p_3))$$

Note that simultaneous substitution is in general not the same as iterated substitution, e.g., compare $\varphi[\theta_1/p_1, \theta_2/p_2]$ above with $(\varphi[\theta_1/p_1])[\theta_2/p_2]$, which is:

$$\begin{aligned} \Diamond(p_2 \rightarrow p_3) &\rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge p_2)[\neg\Box p_1/p_2], \text{ i.e.,} \\ \Diamond(\neg\Box p_1 \rightarrow p_3) &\rightarrow \Box(\Diamond(\neg\Box p_1 \rightarrow p_3) \wedge \neg\Box p_1) \end{aligned}$$

and with $(\varphi[\theta_2/p_2])[\theta_1/p_1]$:

$$\begin{aligned} p_1 &\rightarrow \Box(p_1 \wedge \neg\Box p_1)[\Diamond(p_2 \rightarrow p_3)/p_1], \text{ i.e.,} \\ \Diamond(p_2 \rightarrow p_3) &\rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge \neg\Box\Diamond(p_2 \rightarrow p_3)). \end{aligned}$$

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Bibliography