syn.1  Schemas and Validity

Definition syn.1. A schema is a set of formulas comprising all and only the substitution instances of some modal formula $\chi$, i.e.,

$$\{ \psi : \exists \theta_1, \ldots, \exists \theta_n (\psi = \chi[\theta_1/p_1, \ldots, \theta_n/p_n]) \}.$$ 

The formula $\chi$ is called the characteristic formula of the schema, and it is unique up to a renaming of the propositional variables. A formula $\varphi$ is an instance of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting '\(\varphi\)', '\(\psi\)', \ldots, for the atomic components of $\chi$. So, for instance, the following denote schemas: '\(\varphi\)'', '\(\varphi \rightarrow \Box \varphi\)'', '\(\varphi \rightarrow (\psi \rightarrow \varphi)\)'. They correspond to the characteristic formulas $p$, $p \rightarrow \Box p$, $p \rightarrow (q \rightarrow p)$. The schema '\(\varphi\)' denotes the set of all formulas.

Definition syn.2. A schema is true in a model if and only if all of its instances are; and a schema is valid if and only if it is true in every model.

Proposition syn.3. The following schema K is valid

$$\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi).$$

(K)

Proof. We need to show that all instances of the schema are true at every world in every model. So let $M = (W, R, V)$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that the consequent is true as well. In this case, let $M, w \models \Box (\varphi \rightarrow \psi)$ and $M, w \models \Box \varphi$. We need to show $M, w \models \Box \psi$. So let $w'$ be arbitrary such that $Rww'$. Then by the first assumption $M, w' \models \varphi \rightarrow \psi$ and by the second assumption $M, w' \models \varphi$. It follows that $M, w' \models \psi$. Since $w'$ was arbitrary, $M, w \models \Box \psi$.  

Proposition syn.4. The following schema DUAL is valid

$$\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi.$$  

(DUAL)

Proof. Exercise.  

Problem syn.1. Prove Proposition syn.4.

Proposition syn.5. If $\varphi$ and $\varphi \rightarrow \psi$ are true at a world in a model then so is $\psi$. Hence, the valid formulas are closed under modus ponens.

Proposition syn.6. A formula $\varphi$ is valid iff all its substitution instances are. In other words, a schema is valid iff its characteristic formula is.
Valid Schemas | Invalid Schemas
---|---
□(φ → ψ) → (◇φ → ◇ψ) | □(φ ∨ ψ) → (□φ ∨ □ψ)
◇(φ → ψ) → (□φ → ◇ψ) | (◇φ ∧ ◇ψ) → ◇(φ ∧ ψ)
□(φ ∧ ψ) ↔ (□φ ∧ □ψ) | φ → □φ
□φ → □(ψ → φ) | □◇φ → □φ
¬◇φ → □(φ → ψ) | □□φ → □φ
◇(φ ∨ ψ) ↔ (◇φ ∨ ◇ψ) | □◇φ → △φ.

Table 1: Valid and (or?) invalid schemas.

Proof. The “if” direction is obvious, since φ is a substitution instance of itself.

To prove the “only if” direction, we show the following: Suppose $\mathfrak{M} = \langle W, R, V \rangle$ is a modal model, and $ψ \equiv φ[θ_1/p_1, \ldots, θ_n/p_n]$ is a substitution instance of φ. Define $\mathfrak{M}' = \langle W, R, V' \rangle$ by $V'(p_i) = \{ w : \mathfrak{M}, w \models θ_i \}$. Then $\mathfrak{M}, w \models ψ$ if and only if $\mathfrak{M}', w \models φ$, for any $w \in W$. (We leave the proof as an exercise.)

Now suppose that φ was valid, but some substitution instance ψ of φ was not valid. Then for some $\mathfrak{M} = \langle W, R, V \rangle$ and some $w \in W$, $\mathfrak{M}, w \not\models ψ$. But then $\mathfrak{M}', w \not\models φ$ by the claim, and φ is not valid, a contradiction.

Problem syn.2. Prove the claim in the “only if” part of the proof of Proposition syn.6. (Hint: use induction on φ.)

Note, however, that it is not true that a schema is true in a model iff its characteristic formula is. Of course, the “only if” direction holds: if every instance of φ is true in $\mathfrak{M}$, φ itself is true in $\mathfrak{M}$. But it may happen that φ is false at some world in $\mathfrak{M}$ but some instance of φ is false at some world in $\mathfrak{M}$. For a very simple counterexample consider $p$ in a model with only one world $w$ and $V(p) = \{ w \}$, so that $p$ is true at $w$. But $⊥$ is an instance of $p$, and not true at $w$.

Problem syn.3. Show that none of the following formulas are valid:

D: □p → ◇p;
T: □p → p;
B: p → □◇p;
4: □p → □□p;
5: ◇p → □◇p.

Problem syn.4. Prove that the schemas in the first column of Table 1 are valid and those in the second column are not valid.

Problem syn.5. Decide whether the following schemas are valid or invalid:

1. (◇φ → □ψ) → (□φ → □ψ);

2. $\lozenge (\varphi \rightarrow \psi) \lor \Box (\psi \rightarrow \varphi)$.

**Problem syn.6.** For each of the following schemas find a model $\mathcal{M}$ such that every instance of the formula is true in $\mathcal{M}$:

1. $p \rightarrow \lozenge \lozenge p$;
2. $\lozenge p \rightarrow \Box p$.

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**Bibliography**