syn.1 Schemas and Validity

**Definition syn.1.** A *schema* is a set of formulas comprising all and only the substitution instances of some modal formula $\chi$, i.e.,

$$\{\psi : \exists \theta_1, \ldots, \exists \theta_n (\psi = \chi[\theta_1/p_1, \ldots, \theta_n/p_n])\}.$$  

The formula $\chi$ is called the *characteristic formula* of the schema, and it is unique up to a renaming of the propositional variables. A formula $\phi$ is an instance of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting ‘$\phi$’, ‘$\psi$’, . . . , for the atomic components of $\chi$. So, for instance, the following denote schemas: ‘$\phi$’, ‘$\phi \rightarrow \Box \phi$’, ‘$\phi \rightarrow (\psi \rightarrow \phi)$’. They correspond to the characteristic formulas $p$, $p \rightarrow \Box p$, $p \rightarrow (q \rightarrow p)$. The schema ‘$\phi$’ denotes the set of all formulas.

**Definition syn.2.** A schema is true in a model if and only if all of its instances are; and a schema is valid if and only if it is true in every model.

**Proposition syn.3.** The following schema $K$ is valid

$$\Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi).$$  

(K)

*Proof.* We need to show that all instances of the schema are true at every world in every model. So let $\mathcal{M} = (W, R, V)$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that consequent is true as well. In this case, let $\mathcal{M}, w \models \Box(\phi \rightarrow \psi)$ and $\mathcal{M}, w \models \Box \phi$. We need to show $\mathcal{M} \models \Box \psi$. So let $w'$ be arbitrary such that $Rw'w$. Then by the first assumption $\mathcal{M}, w' \models \phi \rightarrow \psi$ and by the second assumption $\mathcal{M}, w' \models \Box \phi$. It follows that $\mathcal{M}, w' \models \psi$. Since $w'$ was arbitrary, $\mathcal{M}, w \models \Box \psi$. □

**Proposition syn.4.** The following schema DUAL is valid

$$\Diamond \phi \leftrightarrow \neg \Box \neg \phi.$$  

(DUAL)

*Proof.* Exercise. □

**Problem syn.1.** Prove Proposition syn.4.

**Proposition syn.5.** If $\phi$ and $\phi \rightarrow \psi$ are true at a world in a model then so is $\psi$. Hence, the valid formulas are closed under modus ponens.

**Proposition syn.6.** A formula $\phi$ is valid iff all its substitution instances are. In other words, a schema is valid iff its characteristic formula is.
Table 1: Valid and (or?) invalid schemas.

<table>
<thead>
<tr>
<th>Valid Schemas</th>
<th>Invalid Schemas</th>
</tr>
</thead>
<tbody>
<tr>
<td>□(φ → ψ) → (◇φ → ◇ψ)</td>
<td>□(φ ∨ ψ) → (□φ ∨ □ψ)</td>
</tr>
<tr>
<td>◇(φ → ψ) → (□φ → ◇ψ)</td>
<td>(◊φ ∧ ◇ψ) → ◇(φ ∧ ψ)</td>
</tr>
<tr>
<td>□φ → □ψ</td>
<td>φ → □φ</td>
</tr>
<tr>
<td>◇φ → □ψ</td>
<td>□ϕ → □ψ</td>
</tr>
<tr>
<td>◇(φ ∨ ψ) ↔ (◊φ ∨ ◇ψ)</td>
<td>□ ◇ ψ → ◇ψ</td>
</tr>
</tbody>
</table>

Proof. The “if” direction is obvious, since ϕ is a substitution instance of itself.

To prove the “only if” direction, we show the following: Suppose $\mathfrak{M} = \langle W, R, V \rangle$ is a modal model, and $ψ ≡ ϕ[θ_1/p_1, . . . , θ_n/p_n]$ is a substitution instance of $ϕ$. Define $\mathfrak{M}' = \langle W, R, V' \rangle$ by $V'(p_i) = \{ w : \mathfrak{M}, w \models θ_i \}$. Then $\mathfrak{M}, w \models ψ$ iff $\mathfrak{M}', w \models ϕ$, for any $w \in W$. (We leave the proof as an exercise.)

Now suppose that $ϕ$ was valid, but some substitution instance $ψ$ of $ϕ$ was not valid. Then for some $\mathfrak{M} = \langle W, R, V \rangle$ and some $w \in W$, $\mathfrak{M}, w \not\models ψ$. But then $\mathfrak{M}', w \not\models ϕ$ by the claim, and $ϕ$ is not valid, a contradiction.

Problem syn.2. Prove the claim in the “only if” part of the proof of Proposition syn.6. (Hint: use induction on $ϕ$.)

Note, however, that it is not true that a schema is true in a model iff its characteristic formula is. Of course, the “only if” direction holds: if every instance of $ϕ$ is true in $\mathfrak{M}$, $ϕ$ itself is true in $\mathfrak{M}$. But it may happen that $ϕ$ is true in $\mathfrak{M}$ but some instance of $ϕ$ is false at some world in $\mathfrak{M}$. For a very simple counterexample consider $p$ in a model with only one world $w$ and $V(p) = \{ w \}$, so that $p$ is true at $w$. But $⊥$ is an instance of $p$, and not true at $w$.

Problem syn.3. Show that none of the following formulas are valid:

D: □p → ◇p;
T: □p → p;
B: p → □◇p;
4: □p → □□p;
5: ◇p → □◇p.

Problem syn.4. Prove that the schemas in the first column of Table 1 are valid and those in the second column are not valid.

Problem syn.5. Decide whether the following schemas are valid or invalid:

1. (◇φ → □ψ) → (□φ → □ψ);
2. . .
2. $\Diamond(\varphi \rightarrow \psi) \lor □(ψ \rightarrow \varphi)$.

**Problem syn.6.** For each of the following schemas find a model $\mathcal{M}$ such that every instance of the formula is true in $\mathcal{M}$:

1. $p \rightarrow \Diamond \Diamond p$;
2. $\Diamond p \rightarrow □ p$.

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**Bibliography**