syn.1 Schemas and Validity

nml:syn:sch:sec

Definition syn.1. A *schema* is a set of formulas comprising all and only the substitution instances of some modal formula χ, i.e.,

$$\{\psi : \exists\theta_1, \ldots, \exists\theta_n\,(\psi = \chi[\theta_1/p_1, \ldots, \theta_n/p_n])\}.$$

The formula χ is called the *characteristic* formula of the schema, and it is unique up to a renaming of the propositional variables. A formula φ is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting 'φ', 'ψ', …, for the atomic components of χ. So, for instance, the following denote schemas: 'φ', '$\varphi \to \Box\varphi$', '$\varphi \to (\psi \to \varphi)$'. They correspond to the characteristic formulas p, $p \to \Box p$, $p \to (q \to p)$. The schema 'φ' denotes the set of *all* formulas.

Definition syn.2. A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

nml:syn:sch:prop:Kvalid

Proposition syn.3. *The following schema* K *is valid*

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi). \tag{K}$$

Proof. We need to show that all instances of the schema are true at every world in every model. So let $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that consequent is true as well. In this case, let $\mathfrak{M}, w \Vdash \Box(\varphi \to \psi)$ and $\mathfrak{M}, w \Vdash \Box\varphi$. We need to show $\mathfrak{M} \Vdash \Box\psi$. So let w' be arbitrary such that Rww'. Then by the first assumption $\mathfrak{M}, w' \Vdash \varphi \to \psi$ and by the second assumption $\mathfrak{M}, w' \Vdash \varphi$. It follows that $\mathfrak{M}, w' \Vdash \psi$. Since w' was arbitrary, $\mathfrak{M}, w \Vdash \Box\psi$. □

nml:syn:sch:prop:Dual-valid

Proposition syn.4. *The following schema* DUAL *is valid*

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi. \tag{DUAL}$$

Proof. Exercise. □

Problem syn.1. Prove Proposition syn.4.

nml:syn:sch:prop:soundMP

Proposition syn.5. *If φ and $\varphi \to \psi$ are true at a world in a model then so is ψ. Hence, the valid formulas are closed under modus ponens.*

nml:syn:sch:prop:valid-instances

Proposition syn.6. *A formula φ is valid iff all its substitution instances are. In other words, a schema is valid iff its characteristic formula is.*

Valid Schemas	*Invalid Schemas*
$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	$\Box(\varphi \lor \psi) \to (\Box\varphi \lor \Box\psi)$
$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$	$(\Diamond\varphi \land \Diamond\psi) \to \Diamond(\varphi \land \psi)$
$\Box(\varphi \land \psi) \leftrightarrow (\Box\varphi \land \Box\psi)$	$\varphi \to \Box\varphi$
$\Box\varphi \to \Box(\psi \to \varphi)$	$\Box\Diamond\varphi \to \psi$
$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	$\Box\Box\varphi \to \Box\varphi$
$\Diamond(\varphi \lor \psi) \leftrightarrow (\Diamond\varphi \lor \Diamond\psi)$	$\Box\Diamond\varphi \to \Diamond\Box\varphi.$

Table 1: Valid and (or?) invalid schemas.

nml:syn:sch:
tab:valid-invalidSchemas

Proof. The "if" direction is obvious, since φ is a substitution instance of itself.

To prove the "only if" direction, we show the following: Suppose $\mathfrak{M} = \langle W, R, V\rangle$ is a modal model, and $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is a substitution instance of φ. Define $\mathfrak{M}' = \langle W, R, V'\rangle$ by $V'(p_i) = \{w : \mathfrak{M}, w \Vdash \theta_i\}$. Then $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}', w \Vdash \varphi$, for any $w \in W$. (We leave the proof as an exercise.) Now suppose that φ was valid, but some substitution instance ψ of φ was not valid. Then for some $\mathfrak{M} = \langle W, R, V\rangle$ and some $w \in W$, $\mathfrak{M}, w \nVdash \psi$. But then $\mathfrak{M}', w \nVdash \varphi$ by the claim, and φ is not valid, a contradiction. $\Box$

Problem syn.2. Prove the claim in the "only if" part of the proof of Proposition syn.6. (Hint: use induction on φ.)

Note, however, that it is not true that a schema is true in a model iff its characteristic formula is. Of course, the "only if" direction holds: if every instance of φ is true in $\mathfrak{M}$, φ itself is true in $\mathfrak{M}$. But it may happen that φ is true in $\mathfrak{M}$ but some instance of φ is false at some world in $\mathfrak{M}$. For a very simple counterexample consider p in a model with only one world w and $V(p) = \{w\}$, so that p is true at w. But $\bot$ is an instance of p, and not true at w.

Problem syn.3. Show that none of the following formulas are valid:

D: $\Box p \to \Diamond p$;

T: $\Box p \to p$;

B: $p \to \Box\Diamond p$;

4: $\Box p \to \Box\Box p$;

5: $\Diamond p \to \Box\Diamond p$.

Problem syn.4. Prove that the schemas in the first column of Table 1 are valid and those in the second column are not valid.

Problem syn.5. Decide whether the following schemas are valid or invalid:

1. $(\Diamond\varphi \to \Box\psi) \to (\Box\varphi \to \Box\psi)$;

2. $\Diamond(\varphi \to \psi) \lor \Box(\psi \to \varphi)$.

Problem syn.6. For each of the following schemas find a model $\mathfrak{M}$ such that every instance of the formula is true in $\mathfrak{M}$:

1. $p \to \Diamond\Diamond p$;
2. $\Diamond p \to \Box p$.

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Bibliography