

## syn.1 Schemas and Validity

mod:syn:sch:  
sec

**Definition syn.1.** A *schema* is a set of **formulas** comprising all and only the substitution instances of some modal **formula**  $\chi$ , i.e.,

$$\{\psi : \exists\theta_1, \dots, \exists\theta_n (\psi = \chi[\theta_1/p_1, \dots, \theta_n/p_n])\}.$$

The **formula**  $\chi$  is called the *characteristic formula* of the schema, and it is unique up to a renaming of the propositional variables. A **formula**  $\varphi$  is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting ' $\varphi$ ', ' $\psi$ ',  $\dots$ , for the atomic components of  $\chi$ . So, for instance, the following denote schemas: ' $\varphi$ ', ' $\varphi \rightarrow \Box\varphi$ ', ' $\varphi \rightarrow (\psi \rightarrow \varphi)$ '. They correspond to the characteristic **formulas**  $p$ ,  $p \rightarrow \Box p$ ,  $p \rightarrow (q \rightarrow p)$ . The schema ' $\varphi$ ' denotes the set of *all formulas*.

**Definition syn.2.** A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

mod:syn:sch:  
prop:Kvalid **Proposition syn.3.** *The following schema K is valid*

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi). \quad (\text{K})$$

*Proof.* We need to show that all instances of the schema are true at every world in every model. So let  $\mathfrak{M} = \langle W, R, V \rangle$  and  $w \in W$  be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that consequent is true as well. In this case, let  $\mathfrak{M}, w \Vdash \Box(\varphi \rightarrow \psi)$  and  $\mathfrak{M}, w \Vdash \Box\varphi$ . We need to show  $\mathfrak{M}, w' \Vdash \Box\psi$ . So let  $w'$  be arbitrary such that  $Rww'$ . Then by the first assumption  $\mathfrak{M}, w' \Vdash \varphi \rightarrow \psi$  and by the second assumption  $\mathfrak{M}, w' \Vdash \varphi$ . It follows that  $\mathfrak{M}, w' \Vdash \psi$ . Since  $w'$  was arbitrary,  $\mathfrak{M}, w \Vdash \Box\psi$ .  $\square$

mod:syn:sch:  
prop:Dual-valid **Proposition syn.4.** *The following schema DUAL is valid*

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi. \quad (\text{DUAL})$$

*Proof.* Exercise.  $\square$

**Problem syn.1.** Prove Proposition syn.4.

mod:syn:sch:  
prop:soundMP **Proposition syn.5.** *If  $\varphi$  and  $\varphi \rightarrow \psi$  are true at a world in a model then so is  $\psi$ . Hence, the valid **formulas** are closed under modus ponens.*

mod:syn:sch:  
prop:valid-instances **Proposition syn.6.** *A **formula**  $\varphi$  is valid iff all its substitution instances are. In other words, a schema is valid iff its characteristic **formula** is.*

Valid Schemas	Invalid Schemas
$\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$	$\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Box\psi)$
$\Diamond(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Diamond\psi)$	$(\Diamond\varphi \wedge \Diamond\psi) \rightarrow \Diamond(\varphi \wedge \psi)$
$\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$	$\varphi \rightarrow \Box\varphi$
$\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$	$\Box\Diamond\varphi \rightarrow \psi$
$\neg\Diamond\varphi \rightarrow \Box(\varphi \rightarrow \psi)$	$\Box\Box\varphi \rightarrow \Box\varphi$
$\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$	$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

Table 1: Valid and (or?) invalid schemas.

mod:syn:sch:  
tab:valid-invalidSchemas

*Proof.* The “if” direction is obvious, since  $\varphi$  is a substitution instance of itself.

To prove the “only if” direction, we show the following: Suppose  $\mathfrak{M} = \langle W, R, V \rangle$  is a modal model, and  $\psi \equiv \varphi[D_1/p_1, \dots, D_n/p_n]$  is a substitution instance of  $\varphi$ . Define  $\mathfrak{M}' = \langle W, R, V' \rangle$  by  $V(p_i) = \{w : \mathfrak{M} \Vdash \theta_i w\}$ . Then  $\mathfrak{M} \Vdash \psi w$  iff  $\mathfrak{M}' \Vdash \varphi w$ , for any  $w \in W$ . (We leave the proof as an exercise.) Now suppose that  $\varphi$  was valid, but some substitution instance  $\psi$  of  $\varphi$  was not valid. Then for some  $\mathfrak{M} = \langle W, R, V \rangle$  and some  $w \in W$ ,  $\mathfrak{M} \not\Vdash \psi w$ . But then  $\mathfrak{M}' \not\Vdash \varphi w$  by the claim, and  $\varphi$  is not valid, a contradiction.  $\square$

**Problem syn.2.** Prove the claim in the “only if” part of the proof of [Proposition syn.6](#). (Hint: use induction on  $\varphi$ .)

Note, however, that it is not true that a schema is true in a model iff its characteristic formula is. Of course, the “only if” direction holds: if every instance of  $\varphi$  is true in  $\mathfrak{M}$ ,  $\varphi$  itself is true in  $\mathfrak{M}$ . But it may happen that  $\varphi$  is true in  $\mathfrak{M}$  but some instance of  $\varphi$  is false at some world in  $\mathfrak{M}$ . For a very simple counterexample consider  $p$  in a model with only one world  $w$  and  $V(p) = \{w\}$ , so that  $p$  is true at  $w$ . But  $\perp$  is an instance of  $p$ , and not true at  $w$ .

**Problem syn.3.** Show that none of the following [formulas](#) are valid:

- D:  $\Box p \rightarrow \Diamond p$ ;
- T:  $\Box p \rightarrow p$ ;
- B:  $p \rightarrow \Box\Diamond p$ ;
- 4:  $\Box p \rightarrow \Box\Box p$ ;
- 5:  $\Diamond p \rightarrow \Box\Diamond p$ .

**Problem syn.4.** Prove that the schemas in the first column of [table 1](#) are valid and those in the second column are not valid.

**Problem syn.5.** Decide whether the following schemas are valid or invalid:

1.  $(\Diamond\varphi \rightarrow \Box\psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ ;
2.  $\Diamond(\varphi \rightarrow \psi) \vee \Box(\psi \rightarrow \varphi)$ .

**Problem syn.6.** For each of the following schemas find a model  $\mathfrak{M}$  such that every instance of the formula is true in  $\mathfrak{M}$ :

1.  $p \rightarrow \Diamond\Diamond p$ ;
2.  $\Diamond p \rightarrow \Box p$ .

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**Bibliography**