syn.1 Schemas

mod:syn:sch:sec

Definition syn.1. A *schema* is a set of formulas comprising all and only the substitution instances of some modal formula χ, i.e.,

$$\{\psi : \exists\theta_1, \dots, \exists\theta_n\, (\psi = \chi[\theta_1/p_1, \dots, \theta_n/p_n])\}.$$

The formula χ is called the *characteristic* formula of the schema, and it is unique up to a renaming of the propositional variables. A formula φ is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting 'φ', 'ψ', ..., for the atomic components of χ. So, for instance, the following denote schemas: 'φ', '$\varphi \to \Box\varphi$', '$\varphi \to (\psi \to \varphi)$', etc. The schema '$\varphi$' denotes the set of *all* formulas. However, we will also use φ as a meta-linguistic variable for schemas themselves.

Definition syn.2. A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

mod:syn:sch:thm:Kvalid

Theorem syn.3. *The following schema* K *is valid*

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi). \tag{K}$$

Proof. We need to show that all instances of the schema are true at every world in every model. So let $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that consequent is true as well. In this case, let $\mathfrak{M}, w \models \Box(\varphi \to \psi)$ and $\mathfrak{M}, w \models \Box\varphi$. We need to show $\mathfrak{M} \models \Box\psi$. So let w' be arbitrary such that Rww'. Then by the first assumption $\mathfrak{M}, w' \models \varphi \to \psi$ and by the second assumption $\mathfrak{M}, w' \models \varphi$. It follows that $\mathfrak{M}, w' \models \psi$. Since w' was arbitrary, $\mathfrak{M}, w \models \Box\psi$. $\Box$

mod:syn:sch:prop:soundMP

Proposition syn.4. *Show that if φ and $\varphi \to \psi$ are true at a world in a model then so is ψ. Hence, the valid formulas are closed under modus ponens.*

Problem syn.1. Show that none of the following schemas are valid:

D: $\Box\varphi \to \Diamond\varphi$;

T: $\Box\varphi \to \varphi$;

B: $\varphi \to \Box\Diamond\varphi$;

4: $\Box\varphi \to \Box\Box\varphi$;

5: $\Diamond\varphi \to \Box\Diamond\varphi$.

Problem syn.2. Prove that the schemas in the first column of Figure 1 are valid and those in the second column are not valid.

Valid Schemas	*Invalid Schemas*
$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	$\Box(\varphi \lor \psi) \to (\Box\varphi \lor \Box\psi)$
$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$	$(\Diamond\varphi \land \Diamond\psi) \to \Diamond(\varphi \land \psi)$
$\Box(\varphi \land \psi) \leftrightarrow (\Box\varphi \land \Box\psi)$	$\varphi \to \Box\varphi$
$\Box\varphi \to \Box(\psi \to \varphi)$	$\Box\Diamond\varphi \to \psi$
$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	$\Box\Box\varphi \to \Box\varphi$
$\Diamond(\varphi \lor \psi) \leftrightarrow (\Diamond\varphi \lor \Diamond\psi)$	$\Box\Diamond\varphi \to \Diamond\Box\varphi$.

Figure 1: Valid and (or?) invalid schemas.

mod:syn:sch:
fig:valid-invalidSchemas

Problem syn.3. Decide whether the following schemas are valid or invalid:

1. $(\Diamond\varphi \to \Box\psi) \to (\Box\varphi \to \Box\psi)$;

2. $\Diamond(\varphi \to \psi) \lor \Box(\psi \to \varphi)$.

Problem syn.4. For each of the following schemas find a model $\mathfrak{M}$ such that every instance of the schema is true in $\mathfrak{M}$:

1. $\varphi \to \Diamond\Diamond\varphi$;

2. $\Diamond\varphi \to \Box\varphi$.

Photo Credits

Bibliography