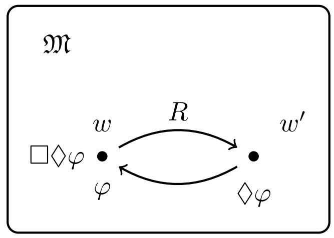

Figure 1: The argument from symmetry.

mod:syn:acc: fig:Bsymm

syn.1 Properties of Accessibility Relations

mod:syn:acc: sec

Definition syn.1. We single out the following five potential properties of an accessibility relation:

R is called ...	*... if it satisfies:*
"serial"	$\forall u \exists v\, Ruv$;
"reflexive"	$\forall w\, Rww$;
"symmetric"	$\forall u \forall v (Ruv \rightarrow Rvu)$;
"transitive"	$\forall u \forall v \forall w (Ruv \land Rvw \rightarrow Ruw)$;
"euclidean"	$\forall w \forall u \forall v (Rwu \land Rwv \rightarrow Ruv)$.

mod:syn:acc: thm:soundschemas

Theorem syn.2. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. Then:*

1. *If R is serial then schema* D*, i.e., $\Box\varphi \rightarrow \Diamond\varphi$, is true in $\mathfrak{M}$;*
2. *If R is reflexive then schema* T*, i.e., $\Box\varphi \rightarrow \varphi$, is true in $\mathfrak{M}$;*
3. *If R is symmetric then schema* B*, i.e., $\varphi \rightarrow \Box\Diamond\varphi$, is true in $\mathfrak{M}$;*
4. *If R is transitive then schema* 4*, i.e., $\Box\varphi \rightarrow \Box\Box\varphi$, is true in $\mathfrak{M}$;*
5. *If R is euclidean then schema* 5*, i.e., $\Diamond\varphi \rightarrow \Box\Diamond\varphi$, is true in $\mathfrak{M}$.*

Proof. Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true all worlds in the model. So let $\varphi \rightarrow \Box\Diamond\varphi$ be a given instance of B, and let $w \in W$ be an arbitrary world. Suppose the antecedent φ is true at w, in order to show that $\Box\Diamond\varphi$ is true at w. So we need to show that $\Diamond\varphi$ is true at all w' accessible from w. Now, for any w' such that Rww' we have, using the hypothesis of symmetry, that also $Rw'w$ (see Figure 1). Since $\mathfrak{M}, w \models \varphi$, we have $\mathfrak{M}, w' \models \Diamond\varphi$. Since w' was an arbitrary world such that Rww', we have $\mathfrak{M}, w \models \Box\Diamond\varphi$. □

Problem syn.1. Complete the proof of Theorem syn.2

Notice that the converse implications of Theorem syn.2 do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property (a counterexample is provided by Example syn.3).

Example syn.3. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model such that $W = \{u, v\}$, where worlds u and v are related by R: i.e., both Ruv and Rvu. Suppose that for all p: $u \in V(p) \Leftrightarrow v \in V(p)$. Then: mod:syn:acc:ex:reflexive

1. For all φ: $\mathfrak{M}, u \models \varphi$ if and only if $\mathfrak{M}, v \models \varphi$ (use induction on φ).

2. Schema T is true in $\mathfrak{M}$.

Since $\mathfrak{M}$ is not reflexive (it is, in fact, *irreflexive*), the converse of Theorem syn.2 fails in the case of T (similar arguments can be given for some—though not all—the other schemas mentioned in Theorem syn.2).

Problem syn.2. Prove the claims in Example syn.3.

Photo Credits

Bibliography