

syn.1 Properties of Accessibility Relations

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sec

Definition syn.1. We single out the following five potential properties of an accessibility relation:

<i>R is called ...</i>	<i>... if it satisfies:</i>
“serial”	$\forall u \exists v Ruv$;
“reflexive”	$\forall w Rww$;
“symmetric”	$\forall u \forall v (Ruv \rightarrow Rvu)$;
“transitive”	$\forall u \forall v \forall w (Ruv \wedge Rvw \rightarrow Ruw)$;
“euclidean”	$\forall w \forall u \forall v (Rwu \wedge Rvw \rightarrow Ruw)$.

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thm:soundschemas

Theorem syn.2. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. Then:

1. If R is serial then every instance of D, i.e., $\Box\varphi \rightarrow \Diamond\varphi$, is true in \mathfrak{M} ;
2. If R is reflexive then every instance of T, i.e., $\Box\varphi \rightarrow \varphi$, is true in \mathfrak{M} ;
3. If R is symmetric then every instance of B, i.e., $\varphi \rightarrow \Box\Diamond\varphi$, is true in \mathfrak{M} ;
4. If R is transitive then every instance of 4, i.e., $\Box\varphi \rightarrow \Box\Box\varphi$, is true in \mathfrak{M} ;
5. If R is euclidean then every instance of 5, i.e., $\Diamond\varphi \rightarrow \Box\Diamond\varphi$, is true in \mathfrak{M} .

Proof. Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true all worlds in the model. So let $\varphi \rightarrow \Box\Diamond\varphi$ be a given instance of B, and let $w \in W$ be an arbitrary world. Suppose the antecedent φ is true at w , in order to show that $\Box\Diamond\varphi$ is true at w . So we need to show that $\Diamond\varphi$ is true at all w' accessible from w . Now, for any w' such that Rww' we have, using the hypothesis of symmetry, that also $Rw'w$ (see Figure 1). Since $\mathfrak{M}, w \models \varphi$, we have $\mathfrak{M}, w' \models \varphi$. Since w' was an arbitrary world such that Rww' , we have $\mathfrak{M}, w \models \Box\Diamond\varphi$. \square

Problem syn.1. Complete the proof of Theorem syn.2

Notice that the converse implications of Theorem syn.2 do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property (a counterexample is provided by Example syn.3).

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ex:reflexive

Example syn.3. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model such that $W = \{u, v\}$, where worlds u and v are related by R : i.e., both Ruv and Rvu . Suppose that for all p : $u \in V(p) \Leftrightarrow v \in V(p)$. Then:

1. For all φ : $\mathfrak{M}, u \models \varphi$ if and only if $\mathfrak{M}, v \models \varphi$ (use induction on φ).
2. Every instance of T is true in \mathfrak{M} .

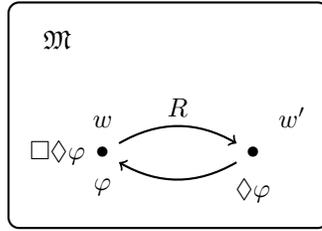


Figure 1: The argument from symmetry.

Since \mathfrak{M} is not reflexive (it is, in fact, *irreflexive*), the converse of [Theorem syn.2](#) fails in the case of \mathbb{T} (similar arguments can be given for some—though not all—the other schemas mentioned in [Theorem syn.2](#)).

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Problem syn.2. Prove the claims in [Example syn.3](#).

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Bibliography