

syn.1 Validity

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sec

Formulas that are true in all models, i.e., true at every world in every model, are particularly interesting. They represent those modal propositions which are true regardless of how \Box and \Diamond are interpreted, as long as the interpretation is “normal” in the sense that it is generated by some accessibility relation on possible worlds. We call such **formulas** *valid*. For instance, $\Box(p \wedge q) \rightarrow \Box p$ is valid. Some **formulas** one might expect to be valid on the basis of the alethic interpretation of \Box , such as $\Box p \rightarrow p$, are not valid, however. Part of the interest of relational models is that different interpretations of \Box and \Diamond can be captured by different kinds of accessibility relations. This suggests that we should define validity not just relative to *all* models, but relative to all models *of a certain kind*. It will turn out, e.g., that $\Box p \rightarrow p$ is true in all models where every world is accessible from itself, i.e., R is reflexive. Defining validity relative to classes of models enables us to formulate this succinctly: $\Box p \rightarrow p$ is valid in the class of reflexive models.

explanation

Definition syn.1. A formula φ is *valid* in a class \mathcal{C} of models if it is true in every model in \mathcal{C} (i.e., true at every world in every model in \mathcal{C}). If φ is valid in \mathcal{C} , we write $\mathcal{C} \models \varphi$, and we write $\models \varphi$ if φ is valid in the class of *all* models.

Proposition syn.2. If φ is valid in \mathcal{C} it is also valid in each class $\mathcal{C}' \subseteq \mathcal{C}$.

mod:syn:val:
prop:Nec-rule

Proposition syn.3. If φ is valid, then so is $\Box\varphi$.

Proof. Assume $\models \varphi$. To show $\models \Box\varphi$ let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and $w \in W$. If Rww' then $\mathfrak{M}, w' \models \varphi$, since φ is valid, and so also $\mathfrak{M}, w \models \Box\varphi$. Since \mathfrak{M} and w were arbitrary, $\models \Box\varphi$. \square

Problem syn.1. Show that the following are valid:

1. $\models \Box p \rightarrow \Box(q \rightarrow p)$;
2. $\models \Box \neg \perp$;
3. $\models \Box p \rightarrow (\Box q \rightarrow \Box p)$.

Problem syn.2. Show that $\varphi \rightarrow \Box\varphi$ is valid in the class \mathcal{C} of models $\mathfrak{M} = \langle W, R, V \rangle$ where $W = \{w\}$. Similarly, show that $\psi \rightarrow \Box\varphi$ and $\Diamond\varphi \rightarrow \psi$ are valid in the class of models $\mathfrak{M} = \langle W, R, V \rangle$ where $R = \emptyset$.

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Bibliography