syn.1 Validity

nml:syn:val: sec

Formulas that are true in all models, i.e., true at every world in every model, are particularly interesting. They represent those modal propositions which are true regardless of how $\Box$ and $\Diamond$ are interpreted, as long as the interpretation is "normal" in the sense that it is generated by some accessibility relation on possible worlds. We call such formulas *valid.* For instance, $\Box(p \land q) \to \Box p$ is valid. Some formulas one might expect to be valid on the basis of the alethic interpretation of $\Box$, such as $\Box p \to p$, are not valid, however. Part of the interest of relational models is that different interpretations of $\Box$ and $\Diamond$ can be captured by different kinds of accessibility relations. This suggests that we should define validity not just relative to *all* models, but relative to all models *of a certain kind.* It will turn out, e.g., that $\Box p \to p$ is true in all models where every world is accessible from itself, i.e., R is reflexive. Defining validity relative to classes of models enables us to formulate this succinctly: $\Box p \to p$ is valid in the class of reflexive models. explanation

Definition syn.1. A formula φ is *valid* in a class $\mathcal{C}$ of models if it is true in every model in $\mathcal{C}$ (i.e., true at every world in every model in $\mathcal{C}$). If φ is valid in $\mathcal{C}$, we write $\mathcal{C} \vDash \varphi$, and we write $\vDash \varphi$ if φ is valid in the class of *all* models.

nml:syn:val: prop:subset-class

Proposition syn.2. *If φ is valid in $\mathcal{C}$ it is also valid in each class $\mathcal{C}' \subseteq \mathcal{C}$.*

nml:syn:val: prop:Nec-rule

Proposition syn.3. *If φ is valid, then so is $\Box\varphi$.*

Proof. Assume $\vDash \varphi$. To show $\vDash \Box\varphi$ let $\mathfrak{M} = \langle W, R, V\rangle$ be a model and $w \in W$. If Rww' then $\mathfrak{M}, w' \Vdash \varphi$, since φ is valid, and so also $\mathfrak{M}, w \Vdash \Box\varphi$. Since $\mathfrak{M}$ and w were arbitrary, $\vDash \Box\varphi$. $\Box$

Problem syn.1. Show that the following are valid:

1. $\vDash \Box p \to \Box(q \to p)$;
2. $\vDash \Box\neg\bot$;
3. $\vDash \Box p \to (\Box q \to \Box p)$.

Problem syn.2. Show that $\varphi \to \Box\varphi$ is valid in the class $\mathcal{C}$ of models $\mathfrak{M} = \langle W, R, V\rangle$ where $W = \{w\}$. Similarly, show that $\psi \to \Box\varphi$ and $\Diamond\varphi \to \psi$ are valid in the class of models $\mathfrak{M} = \langle W, R, V\rangle$ where $R = \emptyset$.

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Bibliography