syn.1 Validity

Formulas that are true in all models, i.e., true at every world in every model, are particularly interesting. They represent those modal propositions which are true regardless of how □ and ♦ are interpreted, as long as the interpretation is “normal” in the sense that it is generated by some accessibility relation on possible worlds. We call such formulas valid. For instance, □(p ∧ q) → □p is valid. Some formulas one might expect to be valid on the basis of the alethic interpretation of □, such as □p → p, are not valid, however. Part of the interest of relational models is that different interpretations of □ and ♦ can be captured by different kinds of accessibility relations. This suggests that we should define validity not just relative to all models, but relative to all models of a certain kind. It will turn out, e.g., that □p → p is true in all models where every world is accessible from itself, i.e., R is reflexive. Defining validity relative to classes of models enables us to formulate this succinctly: □p → p is valid in the class of reflexive models.

Definition syn.1. A formula φ is valid in a class C of models if it is true in every model in C (i.e., true at every world in every model in C). If φ is valid in C, we write C ⊨ φ, and we write ⊨ φ if φ is valid in the class of all models.

Proposition syn.2. If φ is valid in C it is also valid in each class C' ⊆ C.

Proposition syn.3. If φ is valid, then so is □φ.

Proof. Assume ⊨ φ. To show ⊨ □φ let M = ⟨W, R, V⟩ be a model and w ∈ W. If Rw then M, w' ⊨ φ, since φ is valid, and so also M, w ⊨ □φ. Since M and w were arbitrary, ⊨ □φ. □

Problem syn.1. Show that the following are valid:

1. ⊨ □p → □(q → p);
2. ⊨ □¬⊥;
3. ⊨ □p → (□q → □p).

Problem syn.2. Show that φ → □φ is valid in the class C of models M = ⟨W, R, V⟩ where W = {w}. Similarly, show that ψ → □φ and ♦φ → ψ are valid in the class of models M = ⟨W, R, V⟩ where R = ∅.

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Bibliography