

## syn.1 Introduction

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sec Modal Logic deals with *modal propositions* and the entailment relations among them. Examples of modal propositions are the following:

1. It is necessary that  $2 + 2 = 4$ .
2. It is necessarily possible that it will rain tomorrow.
3. If it is necessarily possible that  $\varphi$  then it is possible that  $\varphi$ .

Possibility and necessity are not the only modalities: other unary connectives are also classified as modalities, for instance, “it ought to be the case that  $\varphi$ ,” “It will be the case that  $\varphi$ ,” “Dana knows that  $\varphi$ ,” or “Dana believes that  $\varphi$ .”

Modal logic makes its first appearance in Aristotle’s *De Interpretatione*: he was the first to notice that necessity implies possibility, but not vice versa; that possibility and necessity are inter-definable; that If  $\varphi \wedge \psi$  is possibly true then  $\varphi$  is possibly true and  $\psi$  is possibly true, but not conversely; and that if  $\varphi \rightarrow \psi$  is necessary, then if  $\varphi$  is necessary, so is  $\psi$ .

The first modern approach to modal logic was the work of C. I. Lewis, culminating with Lewis and Langford, *Symbolic Logic* (1932). Lewis & Langford were unhappy with the representation of implication by means of the material conditional:  $\varphi \rightarrow \psi$  is a poor substitute for “ $\varphi$  implies  $\psi$ .” Instead, they proposed to characterize implication as “Necessarily, if  $\varphi$  then  $\psi$ ,” symbolized as  $\varphi \rightarrow \psi$ . In trying to sort out the different properties, Lewis identified five different modal systems, **S1**, . . . , **S4**, **S5**, the last two of which are still in use.

The approach of Lewis and Langford was purely *syntactical*: they identified reasonable axioms and rules and investigated what was provable with those means. A semantic approach remained elusive for a long time, until a first attempt was made by Rudolf Carnap in *Meaning and Necessity* (1947) using the notion of a *state description*, i.e., a collection of atomic sentences (those that are “true” in that state description). After lifting the truth definition to arbitrary sentences  $\varphi$ , Carnap defines  $\varphi$  to be *necessarily true* if it is true in all state descriptions. Carnap’s approach could not handle *iterated* modalities, in that sentences of the form “Possibly necessarily . . . possibly  $\varphi$ ” always reduce to the innermost modality.

The major breakthrough in modal semantics came with Saul Kripke’s article “A Completeness Theorem in Modal Logic” (JSL 1959). Kripke based his work on Leibniz’s idea that a statement is necessarily true if it is true “at all possible worlds.” This idea, though, suffers from the same drawbacks as Carnap’s, in that the truth of statement at a world  $w$  (or a state description  $s$ ) does not depend on  $w$  at all. So Kripke assumed that worlds are related by an *accessibility relation*  $R$ , and that a statement of the form “Necessarily  $\varphi$ ” is true at a world  $w$  if and only if  $\varphi$  is true at all worlds  $w'$  *accessible from*  $w$ . Semantics that provide some version of this approach are called Kripke semantics and made possible the tumultuous development of modal logics (in the plural).

When interpreted by the Kripke semantics, modal logic shows us what *relational structures* look like “from the inside.” A relational structure is just a set equipped with a binary relation (for instance, the set of students in the class ordered by their social security number is a relational structure). But in fact relational structures come in all sorts of domains: besides relative possibility of states of the world, we can have epistemic states of some agent related by epistemic possibility, or states of a dynamical system with their state transitions, etc. Modal logic can be used to model all of these: the first give us ordinary, alethic, modal logic; the others give us epistemic logic, dynamic logic, etc.

We focus on one particular angle, known to modal logicians as “correspondence theory.” One of the most significant early discoveries of Kripke’s is that many properties of the accessibility relation  $R$  (whether it is transitive, symmetric, etc.) can be characterized *in the modal language* itself by means of appropriate “modal schemas.” Modal logicians say, for instance, that the reflexivity of  $R$  “corresponds” to the schema “If necessarily  $\varphi$ , then  $\varphi$ ”. We explore mainly the correspondence theory of a number of classical systems of modal logic (e.g., **S4** and **S5**) obtained by a combination of the schemas D, T, B, 4, and 5.

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## Bibliography