If R is ...	*... the following schema is true in $\mathfrak{M}$:*
partially functional: $\forall w\forall u\forall v(Rwu \land Rwv \Rightarrow u = v)$	$\Diamond\varphi \to \Box\varphi$;
functional: $\forall u\exists v Ruv$	$\Diamond\varphi \leftrightarrow \Box\varphi$;
weakly dense: $\forall u\forall v(Ruv \Rightarrow \exists w(Ruw \land Rwv))$	$\Box\Box\varphi \to \Box\varphi$;
weakly connected: $\forall w\forall u\forall v((Rwu \land Rwv) \Rightarrow (Ruv \lor u = v \lor Rvu))$	L: $\Box((\varphi \land \Box\varphi) \to \psi) \lor \Box((\psi \land \Box\psi) \to \varphi)$;
weakly directed: $\forall w\forall u\forall v((Rwu \land Rwv) \Rightarrow \exists t(Rut \land Rvt)$	G: $\Diamond\Box\varphi \to \Box\Diamond\varphi$;

Figure 1: Five more correspondence facts.

mod:syn:cor:
fig:anotherfive

syn.1 Frame Correspondence

mod:syn:cor:
sec

Even though the converse implications of ?? fail, they hold if we replace "model" by "frame": for the properties considered in ??, it *is* true that if a schema is valid in a *frame* then the accessibility relation of that frame has the corresponding property. In fact, even more is true in the case of D.

mod:syn:cor:
ex:D.complete.for.models

Example syn.1. Any model where schema D is true is serial.

Problem syn.1. Prove Example syn.1 (Hint: take $\varphi = \lnot\bot$).

Although we will focus on the five classical schemas D, T, B, 4, and 5, we record in Figure 1 a few more correspondences.

Problem syn.2. Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model. Show that if R satisfies the left-hand properties of Figure 1, the corresponding right-hand schemas are true in $\mathfrak{M}$.

We now proceed to establish the full correspondence results for frames. We will consider T, B, 4 and 5, as the case for D already follows from Example syn.1.

mod:syn:cor:
thm:fullCorrespondence

Theorem syn.2. *Recall that a schema* S *is valid in a frame if each of its instances is true in every model based on that frame. Then:*

1. *If* T *is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ is reflexive.*
2. *If* B *is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ is symmetric.*
3. *If* 4 *is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ is transitive.*
4. *If* 5 *is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ is euclidean.*

Proof. The strategy is to devise, for each frame $\mathfrak{F}$, a valuation that will ensure that the frame has the desired property (provided the corresponding schema is true).

1. Suppose T is valid in $\mathfrak{F} = \langle W, R\rangle$, let $w \in W$ be an arbitrary world; we need to show Rww. Fix a propositional variable p and let $u \in V(p)$ if and only if Rwu (when q is other than p, $V(q)$ is arbitrary, say $V(q) = \emptyset$). Let $\mathfrak{M} = \langle W, R, V\rangle$. By construction, for all u such that Rwu: $\mathfrak{M}, u \models p$, and hence $\mathfrak{M}, w \models \Box p$. But by hypothesis $\Box p \to p$, an instance of T, is true at w, so that $\mathfrak{M}, w \models p$, but by definition of V this is possible only if Rww.

2. Suppose B is valid in $\mathfrak{F} = \langle W, R\rangle$, and let $u, v \in W$ be arbitrary worlds such that Ruv; we need to show that Rvu. Fix a propositional variable p, define V such that $w \in V(p)$ if and only if Rvw (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V\rangle$. Notice that the following instance of B: $\neg p \to \Box\Diamond\neg p$, is equivalent to $\Diamond\Box p \to p$. Now, by definition of V, $\mathfrak{M}, w \models p$ for all w such that Rvw, and hence $\mathfrak{M}, v \models \Box p$. Since Ruv, also $\mathfrak{M}, u \models \Diamond\Box p$, and since B is valid in $\mathfrak{F}$, also $\mathfrak{M}, u \models \Diamond\Box p \to p$. It follows that $\mathfrak{M}, u \models p$, whence Rvu, as required.

3. Suppose 4 is valid in $\mathfrak{F} = \langle W, R\rangle$, and let $u, v, w \in W$ be arbitrary worlds such that Ruv and Rvw; we need to show that Ruw. Fix a propositional variable p, define V such that $z \in V(p)$ if and only if Ruz (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V\rangle$. By definition of V, $\mathfrak{M}, z \models p$ for all z such that Ruz, and hence $\mathfrak{M}, u \models \Box p$. But by hypothesis $\Box p \to \Box\Box p$, an instance of 4, is true at u, so that $\mathfrak{M}, u \models \Box\Box p$. Since Ruv and Rvw, we have $\mathfrak{M}, w \models p$, but by definition of V this is possible only if Ruw, as desired.

4. We proceed contrapositively, assuming that the frame $\mathfrak{F} = \langle W, R\rangle$ is not euclidean, and falsifying an instance of 5. Suppose there are worlds u, v, w such that Rwu and Rwv but not Ruv. Fix a propositional variable p and define V such that for all worlds z, $z \in V(p)$ if and only if it is *not* the case that Ruz. Let $\mathfrak{M} = \langle W, R, V\rangle$. Then by hypothesis $\mathfrak{M}, v \models p$ and since Rwv also $\mathfrak{M}, w \models \Diamond p$. However, there is no world y such that Ruy and $\mathfrak{M}, y \models p$ so $\mathfrak{M}, u \models \neg\Diamond p$. Since Rwu, it follows that $\mathfrak{M}, w \not\models \Box\Diamond p$, so that the instance of 5, $\Diamond p \to \Box\Diamond p$ fails at w.

□

Theorem syn.2 also shows that the properties can be combined: for instance if both B and 4 are valid in $\mathfrak{F}$ then the frame is both symmetric and transitive, etc. This is useful because the classical systems **S4** and **S5** are, in fact, just the systems characterized as **KT4** and **KTB4**.

We now record some properties of accessibility relations (in fact, these notions apply to arbitrary binary relations).

Proposition syn.3. *Let R be a binary relation on a set W; then:* mod:syn:cor:prop:relation-facts

1. *If R is reflexive, then it is serial.*
2. *If R is symmetric, then it is transitive if and only if it is euclidean.*

3. *If R is symmetric or euclidean then it is weakly directed (it has the "diamond property").*

4. *If R is euclidean then it is weakly connected.*

5. *If R is functional then it is serial.*

Problem syn.3. Prove Proposition syn.3.

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Bibliography