syn.1 Equivalence Relations and S5

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sec

Definition syn.1. A binary relation R on W is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation R on W is *universal* if and only if Ruv for all $u, v \in W$.

mod:syn:es5: prop:equivalences

Proposition syn.2. *The following are equivalent:*

1. *R is an equivalence relation;*
2. *R is reflexive and euclidean;*
3. *R is serial, symmetric, and transitive;*
4. *R is serial, symmetric, and euclidean.*

Problem syn.1. Prove Proposition syn.2

Proposition syn.2 is the semantic counterpart to **??**, in that it gives equivalent characterization of the modal logic of frames over which R is an equivalence (the logic traditionally referred to as **S5**).

Proposition syn.3. *Let R be an equivalence relation, and for each $w \in W$ define the* equivalence class *of w as the set $[w] = \{w' \in W : Rww'\}$. Then:*

1. *$w \in [w]$;*
2. *R is universal on each equivalence class $[w]$;*
3. *The collection of equivalence classes partitions W into mutually exclusive and jointly exhaustive subsets.*

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Proposition syn.4. *A formula φ is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is an equivalence relation, if and only if it is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is universal. Hence, the logic of universal frames is just* **S5**.

Proof. It's immediate to verify that a universal relation R on W is an equivalence. Hence, if φ is valid in all frames where R is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose ψ is a formula that fails at a world w in a model $\mathfrak{M} = \langle W, R, V\rangle$ based on a frame $\langle W, R\rangle$, where R is an equivalence on W. So $\mathfrak{M}, w \nVdash \psi$. Define a model $\mathfrak{M}' = \langle W', R', V'\rangle$ as follows:

1. $W' = [w]$;
2. R' is universal on W';
3. $V'(p) = V(p) \cap W'$.

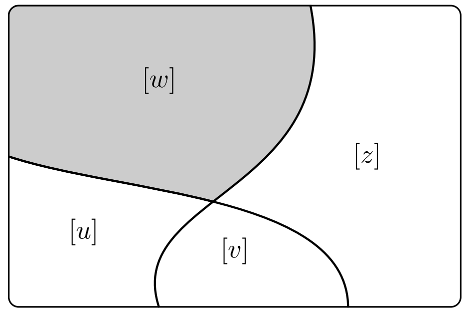

Figure 1: A partition of W in equivalence classes.

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fig:partition

(So the set W' of worlds in $\mathfrak{M}'$ is represented by the shaded area in Figure 1.) It is easy to see that R and R' agree on W'. Then one can show by induction on formulas that for all $w' \in W'$: $\mathfrak{M}', w' \models \varphi$ if and only if $\mathfrak{M}, w' \models \varphi$ for each φ (this makes sense since $W' \subseteq W$). In particular, $\mathfrak{M}', w \not\models \psi$, and ψ fails in a model based on a universal frame. $\square$

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Bibliography