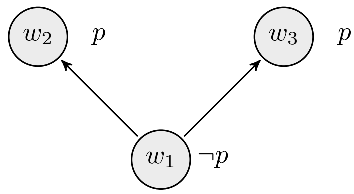

Figure 1: Counterexample to $p \to \Diamond p \vDash \Box p \to p$.

mod:syn:ent:
fig:counterex

syn.1 Entailment

mod:syn:ent:
sec

explanation

With the definition of truth at a world, we can define an entailment relation between formulas. A formula ψ entails φ iff, whenever ψ is true, φ is true as well. Here, “whenever” means both “whichever model we consider” as well as “whichever world in that model we consider.”

Definition syn.1. If Γ is a set of formulas and φ a formula, then Γ *entails* φ, in symbols: $\Gamma \vDash \varphi$, if and only if for every model $\mathfrak{M} = \langle W, R, V \rangle$ and world $w \in W$, if $\mathfrak{M}, w \Vdash \psi$ for every $\psi \in \Gamma$, then $\mathfrak{M}, w \Vdash \varphi$. If Γ contains a single formula ψ, then we write $\psi \vDash \varphi$.

Example syn.2. To show that a formula entails another, we have to reason about all models, using the definition of $\mathfrak{M}, w \Vdash$. For instance, to show $p \to \Diamond p \vDash \Box\neg p \to \neg p$, we might argue as follows: Consider a model $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$, and suppose $\mathfrak{M}, w \Vdash p \to \Diamond p$. We have to show that $\mathfrak{M}, w \Vdash \Box\neg p \to \neg p$. Suppose not. Then $\mathfrak{M}, w \Vdash \Box\neg p$ and $\mathfrak{M}, w \nVdash \neg p$. Since $\mathfrak{M}, w \nVdash \neg p$, $\mathfrak{M}, w \Vdash p$. By assumption, $\mathfrak{M}, w \Vdash p \to \Diamond p$, hence $\mathfrak{M}, w \Vdash \Diamond p$. By definition of $\mathfrak{M}, w \Vdash \Diamond p$, there is some w' with Rww' such that $\mathfrak{M}, w' \Vdash p$. Since also $\mathfrak{M}, w \Vdash \Box\neg p$, $\mathfrak{M}, w' \Vdash \neg p$, a contradiction.

To show that a formula ψ does not entail another φ, we have to give a counterexample, i.e., a model $\mathfrak{M} = \langle W, R, V \rangle$ where we show that at some world $w \in W$, $\mathfrak{M}, w \Vdash \psi$ but $\mathfrak{M}, w \nVdash \varphi$. Let's show that $p \to \Diamond p \nvDash \Box p \to p$. Consider the model in Figure 1. We have $\mathfrak{M}, w_1 \Vdash \Diamond p$ and hence $\mathfrak{M}, w_1 \Vdash p \to \Diamond p$. However, since $\mathfrak{M}, w_1 \Vdash \Box p$ but $\mathfrak{M}, w_1 \nVdash p$, we have $\mathfrak{M}, w_1 \nVdash \Box p \to p$.

Often very simple counterexamples suffice. The model $\mathfrak{M}' = \{W', R', V'\}$ with $W' = \{w\}$, $R' = \emptyset$, and $V'(p) = \emptyset$ is also a counterexample: Since $\mathfrak{M}', w \nVdash p$, $\mathfrak{M}', w \Vdash p \to \Diamond p$. As no worlds are accessible from w, we have $\mathfrak{M}', w \Vdash \Box p$, and so $\mathfrak{M}', w \nVdash \Box p \to p$.

Problem syn.1. Show that $\Box(\varphi \land \psi) \vDash \Box\varphi$.

Problem syn.2. Show that $\Box(p \to q) \nvDash p \to \Box q$ and $p \to \Box q \nvDash \Box(p \to q)$.

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Bibliography

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