Entailment

With the definition of truth at a world, we can define an entailment relation between formulas. A formula $\psi$ entails $\varphi$ iff, whenever $\psi$ is true, $\varphi$ is true as well. Here, “whenever” means both “whichever model we consider” as well as “whichever world in that model we consider.”

**Definition syn.1.** If $\Gamma$ is a set of formulas and $\varphi$ a formula, then $\Gamma$ entails $\varphi$, in symbols: $\Gamma \models \varphi$, if and only if for every model $\mathcal{M} = \langle W, R, V \rangle$ and world $w \in W$, if $\mathcal{M}, w \models \psi$ for every $\psi \in \Gamma$, then $\mathcal{M}, w \models \varphi$. If $\Gamma$ contains a single formula $\psi$, then we write $\psi \models \varphi$.

**Example syn.2.** To show that a formula entails another, we have to reason about all models, using the definition of $\mathcal{M}, w \models \psi$. For instance, to show $p \rightarrow \lozenge p \models \Box \neg p \rightarrow \neg p$, we might argue as follows: Consider a model $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$. Suppose $\mathcal{M}, w \models \lozenge p$, hence $\mathcal{M}, w \models p$. By definition of $\mathcal{M}, w \models \lozenge p$, there is some $w'$ with $Rww'$ such that $\mathcal{M}, w' \models p$. Since also $\mathcal{M}, w \models \Box \neg p$, $\mathcal{M}, w \models \neg p$, a contradiction.

To show that a formula does not entail another $\varphi$, we have to give a counterexample, i.e., a model $\mathcal{M} = \langle W, R, V \rangle$ where we show that at some world $w \in W$, $\mathcal{M}, w \models \psi$ but $\mathcal{M}, w \not\models \varphi$. Let’s show that $p \rightarrow \lozenge p \not\models \Box p \rightarrow p$. Consider the model in Figure 1. We have $\mathcal{M}, w_1 \models \lozenge p$ and hence $\mathcal{M}, w_1 \not\models p \rightarrow \lozenge p$. However, since $\mathcal{M}, w_1 \models \Box p$ but $\mathcal{M}, w_1 \not\models p$, we have $\mathcal{M}, w_1 \not\models \Box \neg p \rightarrow p$.

Often very simple counterexamples suffice. The model $\mathcal{M}' = \langle W', R', V' \rangle$ with $W' = \{ w \}$, $R' = \emptyset$, and $V'(p) = \emptyset$ is also a counterexample: Since $\mathcal{M}', w \not\models p$, $\mathcal{M}', w \not\models p \rightarrow \lozenge p$. As no worlds are accessible from $w$, we have $\mathcal{M}', w \not\models \Box p$, and so $\mathcal{M}', w \not\models \Box \neg p \rightarrow p$.

**Problem syn.1.** Show that $\Box (\varphi \land \psi) \models \Box \varphi$.

**Problem syn.2.** Show that $\Box (p \rightarrow q) \not\models p \rightarrow \Box q$ and $p \rightarrow \Box q \not\models \Box (p \rightarrow q)$.

![Figure 1: Counterexample to $p \rightarrow \lozenge p \models \Box \neg p \rightarrow \neg p$.](fig:counterex)
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Bibliography