Entailment

With the definition of truth at a world, we can define an entailment relation between formulas. A formula ψ entails φ iff, whenever ψ is true, φ is true as well. Here, “whenever” means both “whichever model we consider” as well as “whichever world in that model we consider.”

Definition syn.1. If Γ is a set of formulas and φ a formula, then Γ entails φ, in symbols: Γ |= φ, if and only if for every model M = ⟨W, R, V⟩ and world w ∈ W, if M, w ⊩ ψ for every ψ ∈ Γ, then M, w ⊩ φ. If Γ contains a single formula ψ, then we write ψ |= φ.

Example syn.2. To show that a formula entails another, we have to reason about all models, using the definition of M, w ⊩. For instance, to show p → ◦ p ⊨ ◻ ¬ p → ¬ p, we might argue as follows: Consider a model M = ⟨W, R, V⟩ and w ∈ W, and suppose M, w ⊩ p → ◦ p. We have to show that M, w ⊩ ◻ ¬ p → ¬ p. By assumption, M, w ⊩ p → ◦ p and M, w ⊭ ¬ p. Since M, w ⊭ ¬ p, M, w ⊩ ◻ p. By definition of M, w ⊩ ◻ p, there is some w′ with Rw′w such that M, w′ ⊩ p. Since also M, w ⊩ ◻ ¬ p, M, w′ ⊭ ¬ p, a contradiction.

To show that a formula ψ does not entail another φ, we have to give a counterexample, i.e., a model M = ⟨W, R, V⟩ where we show that at some world w ∈ W, M, w ⊩ ψ but M, w ⊭ φ. Let’s show that p → ◦ p ⊭ ¬ p → p. Consider the model in Figure 1. We have M, w₁ ⊩ ◦ p and hence M, w₁ ⊩ p → ◦ p. However, since M, w₁ ⊩ ◻ p but M, w₁ ⊭ p, we have M, w₁ ⊭ ◻ p → p.

Often very simple counterexamples suffice. The model M′ = {W′, R′, V′} with W′ = {w}, R′ = ∅, and V′(p) = ∅ is also a counterexample: Since M′, w ⊭ p, M′, w ⊩ p → ◦ p. As no worlds are accessible from w, we have M′, w ⊩ ◻ p, and so M′, w ⊭ ◻ p → p.

Problem syn.1. Show that ◻(φ ∧ ψ) |= ◻φ.

Problem syn.2. Show that ◻(p → q) ⊭ p → ◻q and p → ◻q ⊭ ◻(p → q).