Chapter udf

Modal Sequent Calculus

Draft chapter on sequent calculi for modal logic. Needs more examples, soundness and completeness proofs.

seq.1 Introduction

nml:seq:int:sec The sequent calculus for propositional logic can be extended by additional rules that deal with $\Box$ and $\Diamond$. For instance, for **K**, we have **LK** plus:

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\varphi}\,\Box \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Diamond\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}\,\Diamond$$

For extensions of **K**, additional rules have to be added as well.

Not every modal logic has such a sequent calculus. Even **S5**, which is semantically simple (it can be defined without using accessibility relations at all) is not known to have a sequent calculus that results from **LK** which is complete without the rule Cut. However, it has a cut-free complete *hypersequent* calculus.

seq.2 Rules for K

nml:seq:rul:sec The rules for the regular propositional connectives are the same as for regular sequent calculus **LK**. Axioms are also the same: any sequent of the form $\varphi \Rightarrow \varphi$ counts as an axiom.

For the modal operators $\Box$ and $\Diamond$, we have the following additional rules:

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\varphi}\,\Box \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Diamond\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}\,\Diamond$$

Here, $\Box\Gamma$ means the sequence of formulas resulting from Γ by putting $\Box$ in front of every formula in Γ and $\Diamond\Delta$ is the sequence of formulas resulting from Δ by putting $\Diamond$ in front of every formula in Δ. Γ and Δ may be empty; in that case the corresponding part $\Box\Gamma$ and $\Diamond\Delta$ of the conclusion sequent is empty as well.

The restriction of adding a $\Box$ on the right and $\Diamond$ on the left to a single formula φ is necessary. If we allowed to add $\Box$ to any number of formulas on the right or to add $\Diamond$ to any number of formulas on the left we would be able to derive:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\neg\mathrm{R}}{\Rightarrow \Box\varphi, \Box\neg\varphi}\Box*}{\Rightarrow \Box\varphi \lor \Box\neg\varphi}\lor\mathrm{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\neg\mathrm{L}}{\Diamond\neg\varphi, \Diamond\varphi \Rightarrow}\Diamond*}{\Diamond\varphi \Rightarrow \neg\Diamond\neg\varphi}\neg\mathrm{R}}{\Rightarrow \Diamond\varphi \to \neg\Diamond\neg\varphi}\to\mathrm{R}$$

But $\Box\varphi \lor \Box\neg\varphi$ and $\Diamond\varphi \to \neg\Diamond\neg\varphi$ are not valid in **K**.

If we allowed side formulas in addition to φ in the premise, and allowed the $\Box$ rule to add $\Box$ to only φ on the right, or allowed the $\Diamond$ rule to add $\Diamond$ to only φ on the left (but do nothing to the side formulas) we would be able to derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\neg\mathrm{R}}{\Rightarrow \neg\varphi, \varphi}\mathrm{XR}}{\Rightarrow \neg\varphi, \Box\varphi}\Box*}{\Rightarrow \neg\varphi \lor \Box\varphi}\lor\mathrm{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\neg\mathrm{L}}{\Diamond\neg\varphi, \varphi \Rightarrow}\Diamond*}{\varphi \Rightarrow \neg\Diamond\neg\varphi}\neg\mathrm{R}}{\Rightarrow \varphi \to \neg\Diamond\neg\varphi}\to\mathrm{R}$$

But $\neg\varphi \lor \Box\varphi$ (which is equivalent to $\varphi \to \Box\varphi$) and $\varphi \to \neg\Diamond\neg\varphi$ are not valid in **K**.

seq.3 Sequent Derivations for K

nml:seq:prk:sec

Example seq.1. We give a sequent calculus derivation that shows $\vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\psi, \varphi \Rightarrow \varphi} \quad \dfrac{\psi \Rightarrow \psi}{\psi, \varphi \Rightarrow \psi}}{\psi, \varphi \Rightarrow \varphi \land \psi}\land\mathrm{R}}{\Box\psi, \Box\varphi \Rightarrow \Box(\varphi \land \psi)}\Box}{\Box\varphi \land \Box\psi, \Box\varphi \Rightarrow \Box(\varphi \land \psi)}\land\mathrm{L}}{\Box\varphi, \Box\varphi \land \Box\psi \Rightarrow \Box(\varphi \land \psi)}\mathrm{XL}}{\Box\varphi \land \Box\psi, \Box\varphi \land \Box\psi \Rightarrow \Box(\varphi \land \psi)}\land\mathrm{L}}{\Box\varphi \land \Box\psi \Rightarrow \Box(\varphi \land \psi)}\mathrm{CL}}{\Rightarrow (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)}\to\mathrm{R}$$

Example seq.2. We give a sequent calculus derivation that shows $\vdash \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$.

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\doubleLine
\UnaryInfC{$\varphi \Rightarrow \varphi, \psi$}
\AxiomC{$\psi \Rightarrow \psi$}
\doubleLine
\UnaryInfC{$\psi \Rightarrow \varphi, \psi$}
\RightLabel{$\lor$L}
\BinaryInfC{$\varphi \lor \psi \Rightarrow \varphi, \psi$}
\RightLabel{$\Diamond$}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi, \Diamond\psi$}
\RightLabel{$\lor$R}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi, \Diamond\varphi \lor \Diamond\psi$}
\RightLabel{XR}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi \lor \Diamond\psi, \Diamond\varphi$}
\RightLabel{$\lor$R}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi \lor \Diamond\psi, \Diamond\varphi \lor \Diamond\psi$}
\RightLabel{CR}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi \lor \Diamond\psi$}
\RightLabel{$\to$R}
\UnaryInfC{$\Rightarrow \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$}
\end{prooftree}
$$

Here is a derivation of DUAL.

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\lnot$R}
\UnaryInfC{$\lnot\varphi, \varphi \Rightarrow$}
\RightLabel{$\Diamond$}
\UnaryInfC{$\Diamond\lnot\varphi, \Box\varphi \Rightarrow$}
\RightLabel{$\lnot$R}
\UnaryInfC{$\Box\varphi \Rightarrow \lnot\Diamond\lnot\varphi$}
\RightLabel{$\to$R}
\UnaryInfC{$\Rightarrow \Box\varphi \to \lnot\Diamond\lnot\varphi$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\lnot$R}
\UnaryInfC{$\Rightarrow \varphi, \lnot\varphi$}
\RightLabel{XR}
\UnaryInfC{$\Rightarrow \lnot\varphi, \varphi$}
\RightLabel{$\Box$}
\UnaryInfC{$\Rightarrow \Diamond\lnot\varphi, \Box\varphi$}
\RightLabel{XR}
\UnaryInfC{$\Rightarrow \Box\varphi, \Diamond\lnot\varphi$}
\RightLabel{$\lnot$R}
\UnaryInfC{$\lnot\Diamond\lnot\varphi \Rightarrow \Box\varphi$}
\RightLabel{$\to$R}
\UnaryInfC{$\Rightarrow \lnot\Diamond\lnot\varphi \to \Box\varphi$}
\RightLabel{$\land$R}
\BinaryInfC{$\Rightarrow \Box\varphi \leftrightarrow \lnot\Diamond\lnot\varphi$}
\end{prooftree}
$$

Problem seq.1. Find sequent calculus proofs in **K** for the following formulas:

1. $\Box\lnot p \to \Box(p \to q)$
2. $(\Box p \lor \Box q) \to \Box(p \lor q)$
3. $\Diamond p \to \Diamond(p \lor q)$
4. $\Box(p \land q) \to \Box p$

seq.4 Rules for Other Accessibility Relations

nml:seq:mru:sec

In order to deal with logics determined by special accessibility relations, we consider the additional rules in Table seq.1.

Adding these rules results in systems that are sound and complete for the logics given in Table seq.2.

Example seq.3. We give a sequent derivation that shows $\mathbf{K4} \vdash 4$, i.e., $\Box\varphi \to \Box\Box\varphi$.

$$
\begin{prooftree}
\AxiomC{$\Box\varphi \Rightarrow \Box\varphi$}
\RightLabel{$4\Box$}
\UnaryInfC{$\Box\varphi \Rightarrow \Box\Box\varphi$}
\RightLabel{$\to$R}
\UnaryInfC{$\Rightarrow \Box\varphi \to \Box\Box\varphi$}
\end{prooftree}
$$

Example seq.4. We give a sequent derivation that shows $\mathbf{S5} \vdash 5$, i.e., $\Diamond\varphi \to \Box\Diamond\varphi$.

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ \mathrm{T}\Box \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \Diamond\varphi}\ \mathrm{T}\Diamond$$

$$\frac{\Gamma \Rightarrow \Delta}{\Box\Gamma \Rightarrow \Diamond\Delta}\ \mathrm{D}$$

$$\frac{\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Lambda, \varphi}{\Box\Gamma, \Pi \Rightarrow \Delta, \Diamond\Lambda, \Box\varphi}\ \mathrm{B}\Box \qquad \frac{\varphi, \Diamond\Gamma, \Pi \Rightarrow \Box\Lambda, \Delta}{\Diamond\varphi, \Gamma, \Box\Pi \Rightarrow \Lambda, \Diamond\Delta}\ \mathrm{B}\Diamond$$

$$\frac{\Box\Gamma \Rightarrow \Diamond\Delta, \varphi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\varphi}\ 4\Box \qquad \frac{\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}{\Diamond\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}\ 4\Diamond$$

$$\frac{\Box\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Diamond\Lambda, \varphi}{\Box\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Diamond\Lambda, \Box\varphi}\ 5\Box \qquad \frac{\varphi, \Diamond\Gamma, \Box\Pi \Rightarrow \Diamond\Delta, \Box\Lambda}{\Diamond\varphi, \Diamond\Gamma, \Box\Pi \Rightarrow \Diamond\Delta, \Box\Lambda}\ 5\Diamond$$

Table seq.1: More modal rules.

nml:seq:mru:
tab:more-rules

Logic	R is ...	Rules
T = **KT**	reflexive	$\Box$, T$\Box$, T$\Diamond$
D = **KD**	serial	$\Box$, D
K4	transitive	$\Box$, 4$\Box$, 4$\Diamond$
B = **KTB**	reflexive, symmetric	$\Box$, T$\Box$, T$\Diamond$ B$\Box$, B$\Diamond$
S4 = **KT4**	reflexive, transitive	$\Box$, T$\Box$, T$\Diamond$ 4$\Box$, 4$\Diamond$
S5 = **KT5**	reflexive, transitive, euclidean	$\Box$, T$\Box$, T$\Diamond$ 5$\Box$, 5$\Diamond$

Table seq.2: Sequent rules for various modal logics.

nml:seq:mru:
tab:logics-rules

$$\dfrac{\dfrac{\Diamond\varphi \Rightarrow \Diamond\varphi}{\Diamond\varphi \Rightarrow \Box\Diamond\varphi}\,5\Box}{\Rightarrow \Diamond\varphi \to \Box\Diamond\varphi}\,{\to}\mathrm{R}$$

Example seq.5. The sequent calculus for **S5** is not complete without the Cut rule; e.g., $\Diamond\Box\varphi \to \varphi$, which is valid in **S5**, has no proof without Cut. Here is a derivation using Cut:

$$\dfrac{\dfrac{\dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Diamond\Box\varphi \Rightarrow \Box\varphi}\,5\Diamond \qquad \dfrac{\varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \varphi}\,\mathrm{T}\Box}{\Diamond\Box\varphi \Rightarrow \varphi}\,\mathrm{Cut}}{\Rightarrow \Diamond\Box\varphi \to \varphi}\,{\to}\mathrm{R}$$

Problem seq.2. Give sequent derivations that show the following:

1. **KT5** $\vdash$ B;
2. **KT5** $\vdash$ 4;
3. **KDB4** $\vdash$ T;
4. **KB4** $\vdash$ 5;
5. **KB5** $\vdash$ 4;
6. **KT** $\vdash$ D.

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Bibliography