seq.1 Rules for K

nml:seq:rul:sec The rules for the regular propositional connectives are the same as for regular sequent calculus **LK**. Axioms are also the same: any sequent of the form $\varphi \Rightarrow \varphi$ counts as an axiom.

For the modal operators $\Box$ and $\Diamond$, we have the following additional rules:

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\varphi}\,\Box \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Diamond\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}\,\Diamond$$

Here, $\Box\Gamma$ means the sequence of formulas resulting from Γ by putting $\Box$ in front of every formula in Γ and $\Diamond\Delta$ is the sequence of formulas resulting from Δ by putting $\Diamond$ in front of every formula in Δ. Γ and Δ may be empty; in that case the corresponding part $\Box\Gamma$ and $\Diamond\Delta$ of the conclusion sequent is empty as well.

The restriction of adding a $\Box$ on the right and $\Diamond$ on the left to a single formula φ is necessary. If we allowed to add $\Box$ to any number of formulas on the right or to add $\Diamond$ to any number of formulas on the left we would be able to derive:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\,\neg R}{\Rightarrow \Box\varphi, \Box\neg\varphi}\,\Box *}{\Rightarrow \Box\varphi \lor \Box\neg\varphi}\,\lor R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg L}{\Diamond\neg\varphi, \Diamond\varphi \Rightarrow}\,\Diamond *}{\Diamond\varphi \Rightarrow \neg\Diamond\neg\varphi}\,\neg R}{\Rightarrow \Diamond\varphi \to \neg\Diamond\neg\varphi}\,\to R$$

But $\Box\varphi \lor \Box\neg\varphi$ and $\Diamond\varphi \to \neg\Diamond\neg\varphi$ are not valid in **K**.

If we allowed side formulas in addition to φ in the premise, and allowed the $\Box$ rule to add $\Box$ to only φ on the right, or allowed the $\Diamond$ rule to add $\Diamond$ to only φ on the left (but do nothing to the side formulas) we would be able to derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\,\neg R}{\Rightarrow \neg\varphi, \varphi}\,XR}{\Rightarrow \neg\varphi, \Box\varphi}\,\Box *}{\Rightarrow \neg\varphi \lor \Box\varphi}\,\lor R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg L}{\Diamond\neg\varphi, \varphi \Rightarrow}\,\Diamond *}{\varphi \Rightarrow \neg\Diamond\neg\varphi}\,\neg R}{\Rightarrow \varphi \to \neg\Diamond\neg\varphi}\,\to R$$

But $\neg\varphi \lor \Box\varphi$ (which is equivalent to $\varphi \to \Box\varphi$) and $\varphi \to \neg\Diamond\neg\varphi$ are not valid in **K**.

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Bibliography