$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ \text{T}\Box \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \Diamond\varphi}\ \text{T}\Diamond$$

$$\frac{\Gamma \Rightarrow \Delta}{\Box\Gamma \Rightarrow \Diamond\Delta}\ \text{D}$$

$$\frac{\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Lambda, \varphi}{\Box\Gamma, \Pi \Rightarrow \Delta, \Diamond\Lambda, \Box\varphi}\ \text{B}\Box \qquad \frac{\varphi, \Diamond\Gamma, \Pi \Rightarrow \Box\Lambda, \Delta}{\Diamond\varphi, \Gamma, \Box\Pi \Rightarrow \Lambda, \Diamond\Delta}\ \text{B}\Diamond$$

$$\frac{\Box\Gamma \Rightarrow \Diamond\Delta, \varphi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\varphi}\ 4\Box \qquad \frac{\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}{\Diamond\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}\ 4\Diamond$$

$$\frac{\Box\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Diamond\Lambda, \varphi}{\Box\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Diamond\Lambda, \Box\varphi}\ 5\Box \qquad \frac{\varphi, \Diamond\Gamma, \Box\Pi \Rightarrow \Diamond\Delta, \Box\Lambda}{\Diamond\varphi, \Diamond\Gamma, \Box\Pi \Rightarrow \Diamond\Delta, \Box\Lambda}\ 5\Diamond$$

Table 1: More modal rules.

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tab:more-rules

Logic	R is …	Rules
T = **KT**	reflexive	$\Box$, T$\Box$, T$\Diamond$
D = **KD**	serial	$\Box$, D
K4	transitive	$\Box$, 4$\Box$, 4$\Diamond$
B = **KTB**	reflexive, symmetric	$\Box$, T$\Box$, T$\Diamond$ B$\Box$, B$\Diamond$
S4 = **KT4**	reflexive, transitive	$\Box$, T$\Box$, T$\Diamond$ 4$\Box$, 4$\Diamond$
S5 = **KT5**	reflexive, transitive, euclidean	$\Box$, T$\Box$, T$\Diamond$ 5$\Box$, 5$\Diamond$

Table 2: Sequent rules for various modal logics.

nml:seq:mru:
tab:logics-rules

seq.1 Rules for Other Accessibility Relations

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sec

In order to deal with logics determined by special accessibility relations, we consider the additional rules in Table 1.

Adding these rules results in systems that are sound and complete for the logics given in Table 2.

Example seq.1. We give a sequent derivation that shows **K4** $\vdash$ 4, i.e., $\Box\varphi \to \Box\Box\varphi$.

$$\dfrac{\dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Box\varphi \Rightarrow \Box\Box\varphi}\,4\Box}{\Rightarrow \Box\varphi \to \Box\Box\varphi}\,{\to}\mathrm{R}$$

Example seq.2. We give a sequent derivation that shows **S5** $\vdash$ 5, i.e., $\Diamond\varphi \to \Box\Diamond\varphi$.

$$\dfrac{\dfrac{\Diamond\varphi \Rightarrow \Diamond\varphi}{\Diamond\varphi \Rightarrow \Box\Diamond\varphi}\,5\Box}{\Rightarrow \Diamond\varphi \to \Box\Diamond\varphi}\,{\to}\mathrm{R}$$

Example seq.3. The sequent calculus for **S5** is not complete without the Cut rule; e.g., $\Diamond\Box\varphi \to \varphi$, which is valid in **S5**, has no proof without Cut. Here is a derivation using Cut:

$$\dfrac{\dfrac{\dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Diamond\Box\varphi \Rightarrow \Box\varphi}\,5\Diamond \qquad \dfrac{\varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \varphi}\,\mathrm{T}\Box}{\Diamond\Box\varphi \Rightarrow \varphi}\,\mathrm{Cut}}{\Rightarrow \Diamond\Box\varphi \to \varphi}\,{\to}\mathrm{R}$$

Problem seq.1. Give sequent derivations that show the following:

1. **KT5** $\vdash$ B;
2. **KT5** $\vdash$ 4;
3. **KDB4** $\vdash$ T;
4. **KB4** $\vdash$ 5;
5. **KB5** $\vdash$ 4;
6. **KT** $\vdash$ D.

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Bibliography