Part I

Normal Modal Logics

Chapter 1

Syntax and Semantics of Normal Modal Logics

1.1 Introduction

mod:syn:int:sec

Modal Logic deals with *modal propositions* and the entailment relations among them. Examples of modal propositions are the following:

1. It is necessary that $2 + 2 = 4$.
2. It is necessarily possible that it will rain tomorrow.
3. If it is necessarily possible that φ then it is possible that φ.

Possibility and necessity are not the only modalities: other unary connectives are also classified as modalities, for instance, "it ought to be the case that φ," "It will be the case that φ," "Dana knows that φ," or "Dana believes that φ."

Modal logic makes its first appearance in Aristotle's *De Interpretatione*: he was the first to notice that necessity implies possibility, but not vice versa; that possibility and necessity are inter-definable; that If $\varphi \wedge \psi$ is possibly true then φ is possibly true and ψ is possibly true, but not conversely; and that if $\varphi \to \psi$ is necessary, then if φ is necessary, so is ψ.

The first modern approach to modal logic was the work of C. I. Lewis, culminating with Lewis and Langford, *Symbolic Logic* (1932). Lewis & Langford were unhappy with the representation of implication by means of the material conditional: $\varphi \to \psi$ is a poor substitute for "φ implies ψ." Instead, they proposed to characterize implication as "Necessarily, if φ then ψ," symbolized as $\varphi \strictif \psi$. In trying to sort out the different properties, Lewis indentified five different modal systems, **S1**, ..., **S4**, **S5**, the last two of which are still in use.

The approach of Lewis and Langford was purely *syntactical*: they identified reasonable axioms and rules and investigated wat was provable with those means. A semantic approach remained elusive for a long time, until a first attempt was made by Rudolf Carnap in *Meaning and Necessity* (1947) using the notion of a *state description*, i.e., a collection of atomic sentences (those that are "true" in that state description). After lifting the truth definition to arbitrary sentences φ, Carnap defines φ to be *necessarily true* if it is true in all state descriptions. Carnap's approach could not handle *iterated* modalities, in that sentences of the form "Possibly necessarily ... possibly φ" always reduce to the innermost modality.

The major breakthrough in modal semantics came with Saul Kripke's article "A Completeness Theorem in Modal Logic" (JSL 1959). Kripke based his work on Leibniz's idea that a statement is necessarily true if it is true "at all possible worlds." This idea, though, suffers from the same drawbacks as Carnap's, in that the truth of statement at a world w (or a state description s) does not depend on w at all. So Kripke assumed that worlds are related by an *accessibility relation* R, and that a statement of the form "Necessarily φ" is true at a world w if and only if φ is true at all worlds w' *accessible from* w. Semantics that provide some version of this approach are called Kripke semantics and made possible the tumultuous development of modal logics (in the plural).

When interpreted by the Kripke semantics, modal logic shows us what *relational structures* look like "from the inside." A relational structure is just a set equipped with a binary relation (for instance, the set of students in the class ordered by their social security number is a relational structure). But in fact relational structures come in all sorts of domains: besides relative possibility of states of the world, we can have epistemic states of some agent related by epistemic possibility, or states of a dynamical system with their state transitions, etc. Modal logic can be used to model all of these: the first give us ordinary, alethic, modal logic; the others give us epistemic logic, dynamic logic, etc.

We focus on one particular angle, known to modal logicians as "correspondence theory." One of the most significant early discoveries of Kripke's is that many properties of the accessibility relation R (whether it is transitive, symmetric, etc.) can be characterized *in the modal language* itself by means of appropriate "modal schemas." Modal logicians say, for instance, that the reflexivity of R "corresponds" to the schema "If necessarily φ, then φ". We explore mainly the correspondence theory of a number of classical systems of modal logic (e.g., **S4** and **S5**) obtained by a combination of the schemas D, T, B, 4, and 5.

1.2 The Language of Basic Modal Logic

The basic language of modal logic contains a set Var of propositional variables p_1, p_2, ..., the familiar logical connectives $\neg$ ("not"), $\wedge$ ("and"), $\vee$ ("or"), $\rightarrow$, ("if ... then"), the symbols $\top$ (the truth symbol) and $\bot$ (the falsity symbol), as well as the two basic modalities $\Box$ and $\Diamond$. mod:syn:lan:sec

Definition 1.1. *Formulas* of the basic modal language are inductively defined as follows:

1. Every propositional variable p_i is an (atomic) formula.
2. $\top$ is an (atomic) formula
3. $\bot$ is an (atomic) formula.
4. If φ is a formula, so is $\neg\varphi$.
5. If φ and ψ are formulas, so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$.
6. If φ is a formula, so is $\Box\varphi$.
7. Nothing else is a formula.

If a formula φ does not contain $\Box$, we say it is *modal-free*.

$\Diamond A$ abbreviates $\neg\Box\neg\varphi$. So for instance, $\Diamond\Box p \rightarrow \Diamond\Diamond p$ is short for $\neg\Box\neg\Box p \rightarrow \neg\Box\neg\neg\Box\neg p$.

1.3 Simultaneous Substitution

mod:syn:sub:sec

Definition 1.2. Where φ is a modal formula all of whose propositional variables are among $p_1, \dots, p_n$, and $\chi_1, \dots, \chi_n$ are also modal formulas, we define $\varphi[\chi_1/p_1, \dots, \chi_n/p_n]$ as the result of simultaneously substituting each χ_i for p_i in A. Formally, this is a definition by induction on φ:

1. If φ is the propositional variable q, then $\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = \chi_i$ if Q is p_i for some $i \leq n$, and $\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = Q$ otherwise.
2. If $A = \neg\psi$, then $\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = \neg\psi[\chi_1/p_1, \dots, \chi_n/p_n]$.
3. If $\varphi = (\psi \wedge \theta)$, then
$$\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = (\psi[\chi_1/p_1, \dots, \chi_n/p_n] \wedge \theta[\chi_1/p_1, \dots, \chi_n/p_n]).$$
4. If $\varphi = (\psi \vee \theta)$, then
$$\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = (\psi[\chi_1/p_1, \dots, \chi_n/p_n] \vee \theta[\chi_1/p_1, \dots, \chi_n/p_n]).$$
5. If $\varphi = (\psi \rightarrow \theta)$, then
$$\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = (\psi[\chi_1/p_1, \dots, \chi_n/p_n] \rightarrow \theta[\chi_1/p_1, \dots, \chi_n/p_n]).$$
6. If $\varphi = (\psi \leftrightarrow \theta)$, then
$$\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = (\psi[\chi_1/p_1, \dots, \chi_n/p_n] \leftrightarrow \theta[\chi_1/p_1, \dots, \chi_n/p_n]).$$
7. If $\varphi = \Box\psi$, then $\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = \Box\psi[\chi_1/p_1, \dots, \chi_n/p_n]$.
8. If $\varphi = \Diamond\psi$, then $\varphi[\chi_1/p_1, \dots, \chi_n/p_n] = \Diamond\psi[\chi_1/p_1, \dots, \chi_n/p_n]$.

The formula $\varphi[\chi_1/p_1, \dots, \chi_n/p_n]$ is called a *substitution instance* of φ.

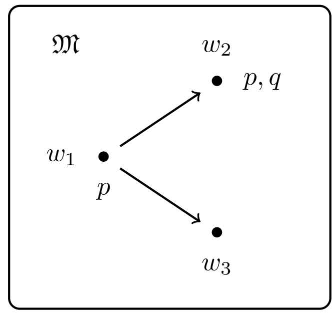


Figure 1.1: A simple model.

mod:syn:rel:
fig:simple

1.4 Relational Models

mod:syn:rel:
sec

Definition 1.3. A *model* for the basic modal language is a triple $\mathfrak{M} = \langle W, R, V \rangle$, where

1. W is a nonempty set of "worlds,"
2. R is a binary accessibility relation on W, and
3. V is a function assigning to each propositional variable p a set $V(p)$ of possible worlds.

The great advantage of relational semantics is that models can be represented by means of simple diagrams, such as the one in Figure 1.1. Worlds are represented by nodes, and world w' is accessible from w precisely when there is an arrow from w to w'. Moreover, we write p next to a world precisely when $w \in V(p)$.

1.5 Truth at a World

mod:syn:trw:
sec

Definition 1.4. *Truth of a formula φ at w* in a $\mathfrak{M}$, $\mathfrak{M}, w \models \varphi$, is defined inductively as follows: mod:syn:trw: defn:mmodels

1. $\mathfrak{M}, w \models p$ iff $w \in V(p)$
2. $\mathfrak{M}, w \models \top$
3. $\mathfrak{M}, w \not\models \bot$
4. $\mathfrak{M}, w \models \neg\psi$ iff $\mathfrak{M}, w \not\models \psi$
5. $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

6. $\mathfrak{M}, w \models \varphi \lor \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$ (or both)

7. $\mathfrak{M}, w \models \varphi \to \psi$ iff $\mathfrak{M}, w \not\models \varphi$ or $\mathfrak{M}, w \models \psi$

mod:syn:trw: defn:sub:mmodels-box
8. $\mathfrak{M}, w \models \Box\varphi$ iff $\mathfrak{M}, w' \models \varphi$ for all $w' \in W$ with wRw'

Note that by clause (8), a formula $\Box\psi$ is satisfied at w whenever there are no w' with wRw'. In such a case $\Box\psi$ is *vacuously* satisfied at w. Also, $\Box\psi$ may be satisifed at w even if ψ is not, and the truth of ψ at w does not guarantee the truth of $\Diamond\psi$ there—this holds if wRw, e.g., if R is reflexive.

Problem 1.1. Consider the model of Figure 1.1. Which of the following hold?

1. $\mathfrak{M}, w_1 \models q$;
2. $\mathfrak{M}, w_3 \models \neg q$;
3. $\mathfrak{M}, w_1 \models p \lor q$;
4. $\mathfrak{M}, w_1 \models \Box(p \lor q)$;
5. $\mathfrak{M}, w_3 \models \Box q$;
6. $\mathfrak{M}, w_3 \models \Box\bot$;
7. $\mathfrak{M}, w_1 \models \Diamond q$;
8. $\mathfrak{M}, w_1 \models \Box q$;
9. $\mathfrak{M}, w_1 \models \neg\Box\Box\neg q$.

Problem 1.2. Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model, and suppose $u, v \in W$ are such that:

1. $u \in V(p)$ if and only if $v \in V(p)$; and
2. for all $w \in W$: Ruw if and only if Rvw.

Using induction on formulas, show that for all formulas φ: $\mathfrak{M}, u \models \varphi$ if and only if $\mathfrak{M}, v \models \varphi$.

Problem 1.3. Let $\mathfrak{M} = \langle M, R, V\rangle$. Show that $\mathfrak{M}, w \models \Diamond\varphi$ if and only if, for some w' with Rww', $\mathfrak{M}, w' \models \varphi]$.

1.6 Entailment

mod:syn:ent: sec

Definition 1.5. If Γ is a set of formulas and φ a formula, then Γ *entails* φ—in symbols: $\Gamma \vDash \varphi$—if and only if for every model $\mathfrak{M} = (W, R, V)$ and world $w \in W$, if $\mathfrak{M}, w \models \psi$ for every $\psi \in \Gamma$, then $\mathfrak{M}, w \models \varphi$. If Γ contains a single formula ψ, then we write $\psi \vDash \varphi$.

Problem 1.4. Show that $\Box(p \to q) \nvDash p \to \Box q$ and $p \to \Box q \nvDash \Box(p \to q)$.

1.7 Truth in a Model

mod:syn:tru: sec

Definition 1.6. A formula φ is *true in a model* $M = \langle W, R, V \rangle$, written $\mathfrak{M} \vDash \varphi$, if and only if $\mathfrak{M}, w \vDash \varphi$ for every $w \in W$.

Proposition 1.7. mod:syn:tru: prop:truthfacts

1. *If* $\mathfrak{M} \vDash \varphi$ *then* $\mathfrak{M} \nvDash \neg\varphi$*, but* not *vice-versa.*

2. *If* $\mathfrak{M} \vDash \varphi \to \psi$ *then* $\mathfrak{M} \vDash \varphi$ *only if* $\mathfrak{M} \vDash \psi$*, but* not *vice-versa.*

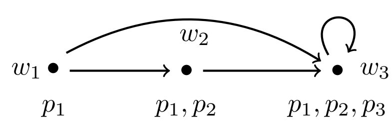
Proof.

1. If $\mathfrak{M} \vDash \varphi$ then φ is true at all worlds in W, and since $W \neq \emptyset$, it can't be that $\mathfrak{M} \vDash \neg\varphi$, or else φ would have to be both true and false at some world. Conversely, if $\mathfrak{M} \nvDash \neg\varphi$ then φ is true at some world $w \in W$; it does not follow that $\mathfrak{M} \vDash \varphi$. For instance, in the model of Figure 1.1, $\mathfrak{M} \nvDash \neg p$, but it does not follow that $\mathfrak{M} \vDash p$.

2. Assume $\mathfrak{M} \vDash \varphi \to \psi$ and $\mathfrak{M} \vDash \varphi$; to show $\mathfrak{M} \vDash \psi$ let $w \in W$ be an arbitrary world. Then $\mathfrak{M}, w \vDash \varphi \to \psi$ and $\mathfrak{M}, w \vDash \psi$, so $\mathfrak{M}, w \vDash \varphi$, and since w was arbitrary, $\mathfrak{M} \vDash \varphi$. The converse fails: we need to find a model $\mathfrak{M}$ such that $\mathfrak{M} \vDash \varphi$ only if $\mathfrak{M} \vDash \psi$, but $\mathfrak{M} \nvDash \varphi \to \psi$. Consider again the model of Figure 1.1: $\mathfrak{M} \nvDash p$ and hence (vacuously) $\mathfrak{M} \vDash p$ only if $\mathfrak{M} \vDash q$. However, $\mathfrak{M} \nvDash p \to q$, as p is true but q false at w_1.

□

Problem 1.5. Consider the following model $\mathfrak{M}$ for the language comprising p_1, p_2, p_3 as the only propositional variables:



Are the following formulas and schemas true in the model $\mathfrak{M}$, i.e., true at every world in $\mathfrak{M}$? Explain.

1. $p \to \Diamond p$ (for p atomic);

2. $\varphi \to \Diamond\varphi$ (for φ arbitrary);

3. $\Box p \to p$ (for p atomic);

4. $\neg p \to \Diamond\Box p$ (for p atomic);

5. $\Diamond\Box\varphi$ (for φ arbitrary);

6. $\Box\Diamond p$ (for p atomic).

1.8 Validity

mod:syn:val: sec

Definition 1.8. A formula φ is *valid* in a class $\mathcal{C}$ of models if it is true in every model in $\mathcal{C}$ (i.e., true at every world in every model in $\mathcal{C}$). If φ is valid in $\mathcal{C}$ we write $\mathcal{C} \vDash \varphi$, and we write $\vDash \varphi$ if φ is valid in the class of *all* models.

Proposition 1.9. *If φ is valid in $\mathcal{C}$ it is also valid in each class $\mathcal{C}' \subseteq \mathcal{C}$.*

mod:syn:val: prop:Nec-rule

Proposition 1.10. *If φ is valid, then so is $\Box\varphi$.*

Proof. Assume $\vDash \varphi$. To show $\vDash \Box\varphi$ let $\mathfrak{M} = \langle W, R, V\rangle$ be a model and $w \in W$. If Rww' then $\mathfrak{M}, w' \models \varphi$, since φ is valid, and so also $\mathfrak{M}, w \models \Box\varphi$. Since $\mathfrak{M}$ and w were arbitrary, $\vDash \Box\varphi$. □

Problem 1.6. Show that the following are valid:

1. $\vDash \Box p \to \Box(q \to p)$;
2. $\vDash \Box\neg\bot$;
3. $\vDash \Box p \to (\Box q \to \Box p)$.

Problem 1.7. Show that $\varphi \to \Box\varphi$ is valid in the class $\mathcal{C}$ of models $\mathfrak{M} = \langle W, R, V\rangle$ where $W = \{w\}$. Similarly, show that $\psi \to \Box\varphi$ and $\Diamond\varphi \to \psi$ are valid in the class of models $\mathfrak{M} = \langle W, R, V\rangle$ where $R = \emptyset$.

1.9 Tautological Instances

mod:syn:tau: sec

Definition 1.11. A modal formula ψ is a *tautological instance* if and only if there is a modal-free tautology φ and formulas $\theta_1, \dots, \theta_n$ such that $\psi = \varphi[\theta_1/p_1, \dots, \theta_n/p_n/]$.

mod:syn:tau: lem:valid-taut

Lemma 1.12. *Suppose φ is a modal-free formula all of whose propositional variables are among $p_1, \dots, p_n$, and let $\theta_1, \dots, \theta_n$ be modal formulas. Then for any assignment v, any model $\mathfrak{M} = \langle W, R, V\rangle$, and any $w \in W$ such that $v(p_i) = 1$ if and only if $\mathfrak{M}, w \models \theta_i$ we have that $v \models \varphi$ if and only if $\mathfrak{M}, w \models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.*

Proof. By induction on φ.

1. φ is atomic: then by the hypothesis it must be some p_i, whence:

$$\begin{aligned} v \models p_i \Leftrightarrow v(p_i) = 1 &\Leftrightarrow \mathfrak{M}, w \models \theta_i \\ &\Leftrightarrow \mathfrak{M}, w \models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]. \end{aligned}$$

2. $\varphi \equiv \neg\psi$:

$$\begin{aligned} v \models \neg\psi &\Leftrightarrow v \not\models \psi && \text{by definition of } v \models \neg\psi; \\ &\Leftrightarrow \mathfrak{M}, w \not\models \psi && \text{by induction hypothesis;} \\ &\Leftrightarrow \mathfrak{M}, w \models \neg\psi && \text{by definition of } v \models \neg\psi. \end{aligned}$$

3. $\varphi \equiv \psi \to \chi$:

$$\begin{aligned} v \models \psi \to \chi &\Leftrightarrow v \not\models \psi \text{ or } v \models \chi && \text{by definition of } v \models \psi \to \chi; \\ &\Leftrightarrow \mathfrak{M}, w \not\models \psi \text{ or } \mathfrak{M}, w \models \chi, && \text{by induction hypothesis;} \\ &\Leftrightarrow \mathfrak{M}, w \models \psi \to \chi && \text{by definition of } \mathfrak{M}, w \models \psi \to \chi. \end{aligned}$$ □

Theorem 1.13. *All tautological instances are valid.* mod:syn:tau: thm:valid-taut

Proof. Contrapositively, suppose φ is such that $\mathfrak{M}, w \not\models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$, for some model $\mathfrak{M}$ and world w. Define an assignment v such that $v(p_i) = 1$ if and only if $\mathfrak{M}, w \models \theta_i$ (and v assigns arbitrary values to $q \notin \{p_1, \dots, p_n\}$). Then by Lemma 1.12, $v \not\models \varphi$, so φ is not a tautology. □

1.10 Schemas

mod:syn:sch: sec

Definition 1.14. A *schema* is a set of formulas comprising all and only the substitution instances of some modal formula χ, i.e.,

$$\{\psi : \exists\theta_1, \dots, \exists\theta_n\,(\psi = \chi[\theta_1/p_1, \dots, \theta_n/p_n])\}.$$

The formula χ is called the *characteristic* formula of the schema, and it is unique up to a renaming of the propositional variables. A formula φ is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting 'φ', 'ψ', …, for the atomic components of χ. So, for instance, the following denote schemas: 'φ', '$\varphi \to \Box\varphi$', '$\varphi \to (\psi \to \varphi)$', etc. The schema '$\varphi$' denotes the set of *all* formulas. However, we will also use φ as a meta-linguistic variable for schemas themselves.

Definition 1.15. A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

Theorem 1.16. *The following schema* K *is valid* mod:syn:sch: thm:Kvalid

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi). \tag{K}$$

Valid Schemas	*Invalid Schemas*
$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	$\Box(\varphi \lor \psi) \to (\Box\varphi \lor \Box\psi)$
$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$	$(\Diamond\varphi \land \Diamond\psi) \to \Diamond(\varphi \land \psi)$
$\Box(\varphi \land \psi) \leftrightarrow (\Box\varphi \land \Box\psi)$	$\varphi \to \Box\varphi$
$\Box\varphi \to \Box(\psi \to \varphi)$	$\Box\Diamond\varphi \to \psi$
$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	$\Box\Box\varphi \to \Box\varphi$
$\Diamond(\varphi \lor \psi) \leftrightarrow (\Diamond\varphi \lor \Diamond\psi)$	$\Box\Diamond\varphi \to \Diamond\Box\varphi.$

Figure 1.2: Valid and (or?) invalid schemas.

mod:syn:sch: fig:valid-invalidSchemas

Proof. We need to show that all instances of the schema are true at every world in every model. So let $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that consequent is true as well. In this case, let $\mathfrak{M}, w \models \Box(\varphi \to \psi)$ and $\mathfrak{M}, w \models \Box\varphi$. We need to show $\mathfrak{M} \models \Box\psi$. So let w' be arbitrary such that Rww'. Then by the first assumption $\mathfrak{M}, w' \models \varphi \to \psi$ and by the second assumption $\mathfrak{M}, w' \models \varphi$. It follows that $\mathfrak{M}, w' \models \psi$. Since w' was arbitrary, $\mathfrak{M}, w \models \Box\psi$. □

mod:syn:sch: prop:soundMP

Proposition 1.17. *Show that if φ and $\varphi \to \psi$ are true at a world in a model then so is ψ. Hence, the valid formulas are closed under modus ponens.*

Problem 1.8. Show that none of the following schemas are valid:

D: $\Box\varphi \to \Diamond\varphi$;

T: $\Box\varphi \to \varphi$;

B: $\varphi \to \Box\Diamond\varphi$;

4: $\Box\varphi \to \Box\Box\varphi$;

5: $\Diamond\varphi \to \Box\Diamond\varphi$.

Problem 1.9. Prove that the schemas in the first column of Figure 1.2 are valid and those in the second column are not valid.

Problem 1.10. Decide whether the following schemas are valid or invalid:

1. $(\Diamond\varphi \to \Box\psi) \to (\Box\varphi \to \Box\psi)$;

2. $\Diamond(\varphi \to \psi) \lor \Box(\psi \to \varphi)$.

Problem 1.11. For each of the following schemas find a model $\mathfrak{M}$ such that every instance of the schema is true in $\mathfrak{M}$:

1. $\varphi \to \Diamond\Diamond\varphi$;

2. $\Diamond\varphi \to \Box\varphi$.

1.11 Frames

mod:syn:fra:sec

Definition 1.18. A *frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a non-empty set of worlds and R a binary relation on W. A model $\mathfrak{M}$ is *based on* a frame $\mathfrak{F} = \langle W, R\rangle$ if and only if $\mathfrak{M} = \langle W, R, V\rangle$.

Definition 1.19. If $\mathcal{F}$ is a class of frames, we write $\mathcal{F} \vDash \varphi$, "$\varphi$ is valid in $\mathcal{F}$," to mean that φ is true in every model $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathcal{F}$.

The reason frames are interesting is that correspondence between schemas and properties of the accessibility relation R is at the level of frames, *not of models.*

Remark 1. Obviously, if a formula or a schema is valid, i.e., valid with respect to the class of *all* models, it is also valid with respect to any class class $\mathcal{F}$ of frames.

1.12 Properties of Accessibility Relations

mod:syn:acc:sec

Definition 1.20. We single out the following five potential properties of an accessibility relation:

R *is called ...*	*... if it satisfies:*
"serial"	$\forall u \exists v\, Ruv$;
"reflexive"	$\forall w\, Rww$;
"symmetric"	$\forall u \forall v (Ruv \rightarrow Rvu)$;
"transitive"	$\forall u \forall v \forall w (Ruv \wedge Rvw \rightarrow Ruw)$;
"euclidean"	$\forall w \forall u \forall v (Rwu \wedge Rwv \rightarrow Ruv)$.

Theorem 1.21. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model. Then:* *mod:syn:acc:thm:soundschemas*

1. *If R is serial then schema* D*, i.e., $\Box\varphi \rightarrow \Diamond\varphi$, is true in $\mathfrak{M}$;*

2. *If R is reflexive then schema* T*, i.e., $\Box\varphi \rightarrow \varphi$, is true in $\mathfrak{M}$;*

3. *If R is symmetric then schema* B*, i.e., $\varphi \rightarrow \Box\Diamond\varphi$, is true in $\mathfrak{M}$;*

4. *If R is transitive then schema* 4*, i.e., $\Box\varphi \rightarrow \Box\Box\varphi$, is true in $\mathfrak{M}$;*

5. *If R is euclidean then schema* 5*, i.e., $\Diamond\varphi \rightarrow \Box\Diamond\varphi$, is true in $\mathfrak{M}$.*

Proof. Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true all worlds in the model. So let $\varphi \rightarrow \Box\Diamond\varphi$ be a given instance of B, and let $w \in W$ be an arbitrary world. Suppose the antecedent φ is true at w, in order to show that $\Box\Diamond\varphi$ is true at w. So we need to show that $\Diamond\varphi$ is true at all w' accessible from w. Now, for

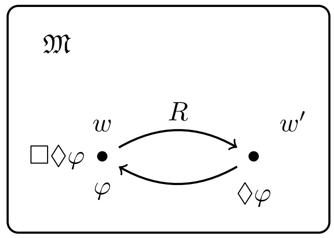


Figure 1.3: The argument from symmetry.

mod:syn:acc: fig:Bsymm

any w' such that Rww' we have, using the hypothesis of symmetry, that also $Rw'w$ (see Figure 1.3). Since $\mathfrak{M}, w \models \varphi$, we have $\mathfrak{M}, w' \models \Diamond\varphi$. Since w' was an arbitrary world such that Rww', we have $\mathfrak{M}, w \models \Box\Diamond\varphi$. □

Problem 1.12. Complete the proof of Theorem 1.21

Notice that the converse implications of Theorem 1.21 do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property (a counterexample is provided by Example 1.22).

mod:syn:acc: ex:reflexive

Example 1.22. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model such that $W = \{u, v\}$, where worlds u and v are related by R: i.e., both Ruv and Rvu. Suppose that for all p: $u \in V(p) \Leftrightarrow v \in V(p)$. Then:

1. For all φ: $\mathfrak{M}, u \models \varphi$ if and only if $\mathfrak{M}, v \models \varphi$ (use induction on φ).

2. Schema T is true in $\mathfrak{M}$.

Since $\mathfrak{M}$ is not reflexive (it is, in fact, *irreflexive*), the converse of Theorem 1.21 fails in the case of T (similar arguments can be given for some—though not all—the other schemas mentioned in Theorem 1.21).

Problem 1.13. Prove the claims in Example 1.22.

1.13 Frame Correspondence

mod:syn:cor: sec

Even though the converse implications of Theorem 1.21 fail, they hold if we replace "model" by "frame": for the properties considered in Theorem 1.21, it *is* true that if a schema is valid in a *frame* then the accessibility relation of that frame has the corresponding property. In fact, even more is true in the case of D.

mod:syn:cor: ex:D.complete.for.models

Example 1.23. Any model where schema D is true is serial.

Problem 1.14. Prove Example 1.23 (Hint: take $\varphi = \neg\bot$).

If R is ...	*... the following schema is true in $\mathfrak{M}$:*
partially functional: $\forall w \forall u \forall v(Rwu \land Rwv \Rightarrow u = v)$	$\Diamond\varphi \to \Box\varphi$;
functional: $\forall u \exists v Ruv$	$\Diamond\varphi \leftrightarrow \Box\varphi$;
weakly dense: $\forall u \forall v(Ruv \Rightarrow \exists w(Ruw \land Rwv))$	$\Box\Box\varphi \to \Box\varphi$;
weakly connected: $\forall w \forall u \forall v((Rwu \land Rwv) \Rightarrow (Ruv \lor u = v \lor Rvu))$	L: $\Box((\varphi \land \Box\varphi) \to \psi) \lor \Box((\psi \land \Box\psi) \to \varphi)$;
weakly directed: $\forall w \forall u \forall v((Rwu \land Rwv) \Rightarrow \exists t(Rut \land Rvt)$	G: $\Diamond\Box\varphi \to \Box\Diamond\varphi$;

Figure 1.4: Five more correspondence facts.

mod:syn:cor:
fig:anotherfive

Although we will focus on the five classical schemas D, T, B, 4, and 5, we record in Figure 1.4 a few more correspondences.

Problem 1.15. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. Show that if R satisfies the left-hand properties of Figure 1.4, the corresponding right-hand schemas are true in $\mathfrak{M}$.

We now proceed to establish the full correspondence results for frames. We will consider T, B, 4 and 5, as the case for D already follows from Example 1.23.

Theorem 1.24. *Recall that a schema* S *is valid in a frame if each of its instances is true in every model based on that frame. Then:* *mod:syn:cor: thm:fullCorrespondence*

1. *If* T *is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ is reflexive.*
2. *If* B *is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ is symmetric.*
3. *If* 4 *is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ is transitive.*
4. *If* 5 *is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ is euclidean.*

Proof. The strategy is to devise, for each frame $\mathfrak{F}$, a valuation that will ensure that the frame has the desired property (provided the corresponding schema is true).

1. Suppose T is valid in $\mathfrak{F} = \langle W, R \rangle$, let $w \in W$ be an arbitrary world; we need to show Rww. Fix a propositional variable p and let $u \in V(p)$ if and only if Rwu (when q is other than p, $V(q)$ is arbitrary, say $V(q) = \emptyset$). Let $\mathfrak{M} = \langle W, R, V \rangle$. By construction, for all u such that Rwu: $\mathfrak{M}, u \models p$, and hence $\mathfrak{M}, w \models \Box p$. But by hypothesis $\Box p \to p$, an instance of T, is true at w, so that $\mathfrak{M}, w \models p$, but by definition of V this is possible only if Rww.

2. Suppose B is valid in $\mathfrak{F} = \langle W, R \rangle$, and let $u, v \in W$ be arbitrary worlds such that Ruv; we need to show that Rvu. Fix a propositional variable p, define V such that $w \in V(p)$ if and only if Rvw (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V \rangle$. Notice that the following instance of B: $\neg p \to \Box\Diamond\neg p$, is equivalent to $\Diamond\Box p \to p$. Now, by definition of V, $\mathfrak{M}, w \models p$ for all w such that Rvw, and hence $\mathfrak{M}, v \models \Box p$. Since Ruv, also $\mathfrak{M}, u \models \Diamond\Box p$, and since B is valid in $\mathfrak{F}$, also $\mathfrak{M}, u \models \Diamond\Box p \to p$. It follows that $\mathfrak{M}, u \models p$, whence Rvu, as required.

3. Suppose 4 is valid in $\mathfrak{F} = \langle W, R \rangle$, and let $u, v, w \in W$ be arbitrary worlds such that Ruv and Rvw; we need to show that Ruw. Fix a propositional variable p, define V such that $z \in V(p)$ if and only if Ruz (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V \rangle$. By definition of V, $\mathfrak{M}, z \models p$ for all z such that Ruz, and hence $\mathfrak{M}, u \models \Box p$. But by hypothesis $\Box p \to \Box\Box p$, an instance of 4, is true at u, so that $\mathfrak{M}, u \models \Box\Box p$. Since Ruv and Rvw, we have $\mathfrak{M}, w \models p$, but by definition of V this is possible only if Ruw, as desired.

4. We proceed contrapositively, assuming that the frame $\mathfrak{F} = \langle W, R \rangle$ is not euclidean, and falsifying an instance of 5. Suppose there are worlds u, v, w such that Rwu and Rwv but not Ruv. Fix a propositional variable p and define V such that for all worlds z, $z \in V(p)$ if and only if it is *not* the case that Ruz. Let $\mathfrak{M} = \langle W, R, V \rangle$. Then by hypothesis $\mathfrak{M}, v \models p$ and since Rwv also $\mathfrak{M}, w \models \Diamond p$. However, there is no world y such that Ruy and $\mathfrak{M}, y \models p$ so $\mathfrak{M}, u \models \neg\Diamond p$. Since Rwu, it follows that $\mathfrak{M}, w \not\models \Box\Diamond p$, so that the instance of 5, $\Diamond p \to \Box\Diamond p$ fails at w.

□

Theorem 1.24 also shows that the properties can be combined: for instance if both B and 4 are valid in $\mathfrak{F}$ then the frame is both symmetric and transitive, etc. This is useful because the classical systems **S4** and **S5** are, in fact, just the systems characterized as **KT4** and **KTB4**.

We now record some properties of accessibility relations (in fact, these notions apply to arbitrary binary relations).

mod:syn:cor: prop:relation-facts

Proposition 1.25. *Let R be a binary relation on a set W; then:*

1. *If R is reflexive, then it is serial.*
2. *If R is symmetric, then it is transitive if and only if it is euclidean.*
3. *If R is symmetric or euclidean then it is weakly directed (it has the "diamond property").*
4. *If R is euclidean then it is weakly connected.*
5. *If R is functional then it is serial.*

Problem 1.16. Prove Proposition 1.25.

1.14 Equivalence Relations and S5

mod:syn:es5: sec

Definition 1.26. A binary relation R on W is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation R on W is *universal* if and only if Ruv for all $u, v \in W$.

Proposition 1.27. *The following are equivalent:* *mod:syn:es5: prop:equivalences*

1. *R is an equivalence relation;*

2. *R is reflexive and euclidean;*

3. *R is serial, symmetric, and transitive;*

4. *R is serial, symmetric, and euclidean.*

Problem 1.17. Prove Proposition 1.27

Proposition 1.27 is the semantic counterpart to Proposition 2.20, in that it gives equivalent characterization of the modal logic of frames over which R is an equivalence (the logic traditionally referred to as **S5**).

Proposition 1.28. *Let R be an equivalence relation, and for each $w \in W$ define the* equivalence class *of w as the set $[w] = \{w' \in W : Rww'\}$. Then:*

1. *$w \in [w]$;*

2. *R is universal on each equivalence class $[w]$;*

3. *The collection of equivalence classes partitions W into mutually exclusive and jointly exhaustive subsets.*

Proposition 1.29. *A formula φ is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is an equivalence relation, if and only if it is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is universal. Hence, the logic of universal frames is just* **S5**. *mod:syn:es5: prop:S5=univ*

Proof. It's immediate to verify that a universal relation R on W is an equivalence. Hence, if φ is valid in all frames where R is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose ψ is a formula that fails at a world w in a model $\mathfrak{M} = \langle W, R, V\rangle$ based on a frame $\langle W, R\rangle$, where R is an equivalence on W. So $\mathfrak{M}, w \nVdash \psi$. Define a model $\mathfrak{M}' = \langle W', R', V'\rangle$ as follows:

1. $W' = [w]$;

2. R' is universal on W';

3. $V'(p) = V(p) \cap W'$.

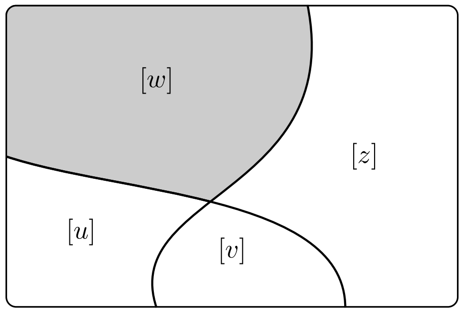


Figure 1.5: A partition of W in equivalence classes.

mod:syn:es5:
fig:partition

(So the set W' of worlds in $\mathfrak{M}'$ is represented by the shaded area in Figure 1.5.) It is easy to see that R and R' agree on W'. Then one can show by induction on formulas that for all $w' \in W'$: $\mathfrak{M}', w' \models \varphi$ if and only if $\mathfrak{M}, w' \models \varphi$ for each φ (this makes sense since $W' \subseteq W$). In particular, $\mathfrak{M}', w \not\models \psi$, and ψ fails in a model based on a universal frame. □

Chapter 2

Axioms, Derivations, and Modal Systems

2.1 Modal Logics

mod:prf:log:sec

Definition 2.1. A *modal logic* is a set Σ of modal formulas which is closed under *tautological implication* in the following sense: if $\varphi_1, \dots, \varphi_n \in \Sigma$ and $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \varphi) \cdots)$ is a tautological instance, then $\varphi \in \Sigma$.

Proposition 2.2. *Every modal logic is closed under the rule of* Modus Ponens*:*

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}\ \text{MP}$$

Proof. $(\varphi \to \psi) \to (\varphi \to \psi)$ is tautological instance, hence if $\varphi \to \psi$ and φ are in Σ, so is ψ. $\square$

2.2 Normal Modal Logics

mod:prf:nor:sec

Definition 2.3. A modal logic Σ is *normal* if it is closed under the rule RK:

$$\frac{\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \cdots)}{\Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n) \cdots)}\ \text{RK}$$

Observe that while tautological implication is "fine-grained" enough to preserve *truth at a world*, the rule RK only preserves *truth in a model* (and hence also validity in a frame or in a class of frames).

Proposition 2.4. *Every normal modal logic Σ is closed under the rule of* Necessitation*:*

$$\frac{\varphi}{\Box\varphi}\ \text{NEC}$$

Proof. NEC is just the special case of RK when $n = 1$. □

Proposition 2.5. *Every normal modal logic Σ contains every instance of* K.

Proof. In fact, K follows from rule RK: $(\varphi \to \psi) \to (\varphi \to \psi)$ is in Σ since it is a tautological instance; one application of RK gives that $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is in Σ as well. □

Proposition 2.6. *Every normal modal logic Σ contains $\neg\Diamond\bot$.* mod:prf:nor:prop:notDiamondBot

Problem 2.1. Prove Proposition 2.6.

2.3 Modal Systems

mod:prf:sys:sec

Proposition 2.7. *Let $\varphi_1, \dots, \varphi_n$ be schemas. Then there is a smallest modal logic Σ containing all instances of $\varphi_1, \dots, \varphi_n$. Such a modal logic is called a* modal system *and denoted by $\mathbf{K}\varphi_1 \dots \varphi_n$. The smallest normal modal logic is denoted by* $\mathbf{K}$.

Proof. Given $\varphi_1, \dots, \varphi_n$, define Σ as the intersection of all normal modal logics containing all instances of $\varphi_1, \dots, \varphi_n$. The intersection is non-empty as $\mathrm{Frm}(\mathcal{L})$, the set of all formulas, is such a modal logic. □

2.4 Logics Defined by Proofs

mod:prf:prf:sec

Definition 2.8. Given a modal system $\mathbf{K}\varphi_1 \dots \varphi_n$ and a formula ψ we say that ψ is *derivable* in $\mathbf{K}\varphi_1 \dots \varphi_n$, written $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$, if and only if there are formulas $\chi_1, \dots, \chi_k$ such that $\chi_k = \psi$ and each χ_i is either a tautological instance, or an instance of the schemas K, $\varphi_1, \dots, \varphi_n$, or it follows from previous formulas by means of the rules MP or NEC.

The following proposition allows us to show that $\psi \in \Sigma$ by exhibiting a Σ-proof of ψ.

Proposition 2.9. $\mathbf{K}\varphi_1 \dots \varphi_n = \{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\}$.

Proof. We use induction on the length of proofs to show that $\{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\} \subseteq \mathbf{K}\varphi_1 \dots \varphi_n$. The converse inclusion follows by showing that $\{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \varphi\}$ is a normal modal logic containing all the instances of the schemas $\varphi_1, \dots, \varphi_n$, and the observation that $\mathbf{K}\varphi_1 \dots \varphi_n$ is, by definition, the smallest such logic. □

2.5 Proofs in K

mod:prf:prk: sec

In order to practice proofs in the smallest modal system, we show the valid formulas on the left-hand side of the table of Figure 1.2 can all be given **K**-proofs. Justifications for steps that are either tautological instances or follow by tautological implication from previous one are just marked "PL" (for "Propositional Logic").

Proposition 2.10. $\mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$

Proof.

1. $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ — PL
2. $\Box[(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ — NEC
3. $\Box(\varphi \to \psi) \to \Box(\neg\psi \to \neg\varphi)$ — K, MP
4. $\Box(\neg\psi \to \neg\varphi) \to (\Box\neg\psi \to \Box\neg\varphi)$ — K
5. $\Box(\varphi \to \psi) \to (\Box\neg\psi \to \Box\neg\varphi)$ — PL, 3,4
6. $(\Box\neg\psi \to \Box\neg\varphi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$ — PL
7. $\Box(\varphi \to \psi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$ — PL, 5, 6
8. $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$ — re-writing.

□

Proposition 2.11. $\mathbf{K} \vdash \Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$

Proof.

1. $\varphi \to (\neg\psi \to \neg(\varphi \to \psi))$ — PL
2. $\Box[\varphi \to (\neg\psi \to \neg(\varphi \to \psi))]$ — NEC
3. $\Box\varphi \to \Box(\neg\psi \to \neg(\varphi \to \psi))$ — K
4. $\Box\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \to \psi))$ — K
5. $\Box\varphi \to (\neg\Box\neg(\varphi \to \psi) \to \neg\Box\neg\psi)$ — PL
6. $\Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$ — re-writing.

□

Proposition 2.12. $\mathbf{K} \vdash \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$

Proof.

1. $\neg(\varphi \to \neg\psi) \to \varphi$ — PL
2. $\Box\neg(\varphi \to \neg\psi) \to \Box\varphi$ — NEC, K
3. $\Box(\varphi \wedge \psi) \to \Box\varphi$ — re-writing
4. $\Box(\varphi \wedge \psi) \to \Box\psi$ — similarly
5. $\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$ — PL, 3,4.

□

Proposition 2.13. $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

Proof.

1.	$\varphi \to (\psi \to (\varphi \land \psi))$	PL
2.	$\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi))$	NEC, K
3.	$(\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$	PL, 2

□

Proposition 2.14. $\mathbf{K} \vdash \Box\varphi \to \Box(\psi \to \varphi)$

Proof.

1.	$\varphi \to (\psi \to \varphi)$	PL
2.	$\Box\varphi \to \Box(\psi \to \varphi)$	NEC, K.

□

Proposition 2.15. $\mathbf{K} \vdash \neg\Diamond\varphi \to \Box(\varphi \to \psi)$

Proof.

1.	$\neg\varphi \to (\varphi \to \psi)$	PL
2.	$\Box\neg\varphi \to \Box(\varphi \to \psi)$	NEC, K
3.	$\neg\neg\Box\neg\varphi \to \Box(\varphi \to \psi)$	PL
4.	$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	re-writing.

□

Proposition 2.16. $\mathbf{K} \vdash (\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi)$

Proof.

1.	$\neg(\neg\varphi \to \psi) \to \neg\varphi$	PL
2.	$\neg(\varphi \lor \psi) \to \neg\varphi$	PL, 1
3.	$\Box\neg(\varphi \lor \psi) \to \Box\neg\varphi$	NEC, K
4.	$\neg\Box\neg\varphi \to \neg\Box\neg(\varphi \lor \psi)$	PL
5.	$\Diamond\varphi \to \Diamond(\varphi \lor \psi)$	re-writing
6.	$\Diamond\psi \to \Diamond(\varphi \lor \psi)$	similarly
7.	$(\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi)$	PL, 5, 6.

□

Proposition 2.17. $\mathbf{K} \vdash \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$

Proof.

1.	$\neg\varphi \to (\neg\psi \to \neg(\varphi \lor \psi)$	PL
2.	$\Box\neg\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \lor \psi)$	NEC, K
3.	$\Box\neg\varphi \to (\neg\Box\neg(\varphi \lor \psi) \to \neg\Box\neg\psi))$	PL, 2
4.	$\neg\Box\neg(\varphi \lor \psi) \to (\Box\neg\varphi \to \neg\Box\neg\psi)$	PL
5.	$\neg\Box\neg(\varphi \lor \psi) \to (\neg\neg\Box\neg\psi \to \neg\Box\neg\varphi)$	PL
6.	$\Diamond(\varphi \lor \psi) \to (\neg\Diamond\psi \to \Diamond\varphi)$	re-writing
7.	$\Diamond(\varphi \lor \psi) \to (\Diamond\psi \lor \Diamond\varphi)$	PL.

□

Problem 2.2. Provide **K**-proofs of the following:

1. $\Diamond\neg\bot \to (\Box\varphi \to \Diamond\varphi)$;
2. $\Box(\varphi \lor \psi) \to (\Diamond\varphi \lor \Box\psi)$;
3. $(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$.

2.6 Dual Schemas

mod:prf:dua: sec

mod:prf:dua: def:duals **Definition 2.18.** Each of the schemas T, B, 4, and 5 has a *dual*, denoted by a subscripted diamond, as follows:

$$\begin{array}{ll} \mathrm{T}_\Diamond: & \varphi \to \Diamond\varphi \\ \mathrm{B}_\Diamond: & \Diamond\Box\varphi \to \varphi \\ 4_\Diamond: & \Diamond\Diamond\varphi \to \Diamond\varphi \\ 5_\Diamond: & \Diamond\Box\varphi \to \Box\varphi \end{array}$$

Each of the dual above schemas is obtained from the corresponding schema by replacing $\neg\varphi$ for φ, contraposing, and re-writing. Schema D is its own dual (modulo the replacement of $\neg\Diamond\neg$ by $\Box$).

Problem 2.3. Show that for each schema φ in Definition 2.18: $\mathbf{K} \vdash \varphi \leftrightarrow \varphi_\Diamond$.

2.7 Proofs in Modal Systems

mod:prf:prs: sec

We now come to proofs in systems of modal logic other than **K**.

mod:prf:prs: prop:S5facts **Proposition 2.19.** *The following provability results obtain:*

1. **KT5** $\vdash$ B;
2. **KT5** $\vdash$ 4;
3. **KDB4** $\vdash$ T;
4. **KB4** $\vdash$ 5;
5. **KB5** $\vdash$ 4;
6. **KT** $\vdash$ D. *mod:prf:prs: prop:S5facts-KT-D*

Proof. We exhibit proofs for each.

1. **KT5** $\vdash$ B:

1.	$\Diamond\varphi \to \Box\Diamond\varphi$	5
2.	$\varphi \to \Diamond\varphi$	T$\Diamond$
3.	$\varphi \to \Box\Diamond\varphi$	PL.

2. **KT5** $\vdash$ 4:

1.	$\Diamond\Box\varphi \to \Box\Diamond\Box\varphi$	5 with $\Box\varphi$ for φ
2.	$\Box\varphi \to \Diamond\Box\varphi$	$\mathrm{T}_\Diamond$ with $\Box\varphi$ for φ
3.	$\Box\varphi \to \Box\Diamond\Box\varphi$	PL, 1, 2
4.	$\Diamond\Box\varphi \to \Box\varphi$	$5_\Diamond$
5.	$\Box\Diamond\Box\varphi \to \Box\Box\varphi$	RK, 4
6.	$\Box\varphi \to \Box\Box\varphi$	PL, 3, 5.

3. **KDB4** $\vdash$ T:

1.	$\Diamond\Box\varphi \to \varphi$	$\mathrm{B}_\Diamond$
2.	$\Box\Box\varphi \to \Diamond\Box\varphi$	D with $\Box\varphi$ for φ
3.	$\Box\Box\varphi \to \varphi$	PL1, 2
4.	$\Box\varphi \to \Box\Box\varphi$	4
5.	$\Box\varphi \to \varphi$	PL, 1, 4.

4. **KB4** $\vdash$ 5:

1.	$\Diamond\varphi \to \Box\Diamond\Diamond\varphi$	B with $\Diamond\varphi$ for φ
2.	$\Diamond\Diamond\varphi \to \Diamond\varphi$	$4_\Diamond$
3.	$\Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi$	RK, 2
4.	$\Diamond\varphi \to \Box\Diamond\varphi$	PL, 1, 3.

5. **KB5** $\vdash$ 4:

1.	$\Box\varphi \to \Box\Diamond\Box\varphi$	B with $\Box\varphi$ for φ
2.	$\Diamond\Box\varphi \to \Box\varphi$	$5_\Diamond$
3.	$\Box\Diamond\Box\varphi \to \Box\Box\varphi$	RK, 2
4.	$\Box\varphi \to \Box\Box\varphi$	PL, 1, 3.

6. **KT** $\vdash$ D:

1.	$\Box\varphi \to \varphi$	T
2.	$\varphi \to \Diamond\varphi$	$\mathrm{T}_\Diamond$
3.	$\Box\varphi \to \Diamond\varphi$	PL, 1, 2

□

Proposition 2.20. **KTB4** = **KT5** = **KDB4** = **KDB5**. *mod:prf:prs: prop:S5*

Problem 2.4. Prove Proposition 2.20.

Definition 2.21. Following tradition, we define **S4** to be the system **KT4**, and **S5** the system **KTB4**.

Proposition 2.20 shows that the classical system **S5** has several equivalent axiomatizations (see Proposition 1.27).

2.8 Soundness

mod:prf:snd:sec

mod:prf:snd:thm:soundness

Theorem 2.22 (Soundness Theorem)**.** *If schemas $\varphi_1, \ldots, \varphi_n$ are valid in the classes of models $\mathcal{C}_n, \ldots, \mathcal{C}_n$, respectively, then $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$ implies that ψ is valid in the class of models $\mathcal{C}_n \cap \cdots \cap \mathcal{C}_n$.*

Proof. By induction on length of proofs. For brevity, put $\mathcal{C} = \mathcal{C}_n \cap \cdots \cap \mathcal{C}_n$.

1. Induction Basis: If ψ has a proof of length 1, then it is either a tautological instance or an instance of K, or an instance of one of the schemas. In the first case, ψ is valid in $\mathcal{C}$, since tautological instance are valid in *any* class of models, by Theorem 1.13. Similarly in the second case, by Theorem 1.16. Finally in the third case, since ψ is valid in $\mathcal{C}_i$ and $\mathcal{C} \subseteq \mathcal{C}_i$, we have that ψ is valid in $\mathcal{C}$ as well.

2. Inductive step: Suppose ψ has a proof of length $k > 1$. If ψ is a tautological instance or an instance of one of the schemas, we proceed as in the previous step. So suppose ψ is obtained by MP from previous formulas $\chi \to \psi$ and χ. Then $\chi \to \psi$ and χ have proofs of length $< k$, and by inductive hypothesis they are valid in $\mathcal{C}$. By Proposition 1.17, ψ is valid in $\mathcal{C}$ as well. Finally suppose ψ is obtained by NEC from χ (so that $\psi = \Box\chi$). By inductive hypothesis, χ is valid in $\mathcal{C}$, and by Proposition 1.10 so is φ. □

2.9 Showing Systems are Distinct

mod:prf:dis:sec

In section 2.7 we saw how to prove that two systems of modal logic are in fact the same system. Theorem 2.22 allows us to show that two modal systems Σ and Σ' are distinct, by finding a formula φ such that $\Sigma' \vdash \varphi$ that fails in a model of Σ.

Proposition 2.23. $\mathbf{KD} \subsetneq \mathbf{KT}$

Proof. This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show $\mathbf{KD} \subseteq \mathbf{KT}$ we need to see that $\mathbf{KD} \vdash \psi$ implies $\mathbf{KT} \vdash \psi$, which follows from $\mathbf{KT} \vdash \mathbf{D}$, as shown in Proposition 2.19(6). To show that the inclusion is proper, by Soundness (Theorem 2.22), it suffices to exhibit a model of **KD** where some instance $\Box\varphi \to \varphi$ of T fails (an easy task left as an exercise), for then by Soundness $\mathbf{KD} \nvdash \Box\varphi \to \varphi$. □

Proposition 2.24. $\mathbf{KB} \neq \mathbf{K4}$.

Proof. We construct a symmetric model where some instance of 4 fails; since obviously the instance is derivable for **K4** but not in **KB**, it will follow $\mathbf{K4} \nsubseteq \mathbf{KB}$. Consider the symmetric model $\mathfrak{M}$ of Figure 2.1 Since the model is symmetric, K and B are true in $\mathfrak{M}$ (by Theorem 1.16 and Theorem 1.21, respectively). However, $\mathfrak{M}, w \nvDash \Box p \to \Box\Box p$. □

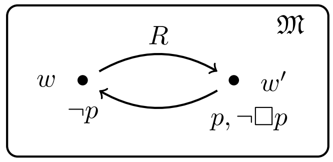


Figure 2.1: A symmetric model falsifying an instance of 4.

mod:prf:dis: fig:Bnot4

Theorem 2.25. $\mathbf{KTB} \nvdash \mathbf{4}$ *and* $\mathbf{KTB} \nvdash \mathbf{5}$. mod:prf:dis: thm:KTBnot45

Proof. By Theorem 1.21 we know that all instances of T and B are true in each reflexive symmetric model (respectively). So by Soundness it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figure Figure 2.2. Then $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$, so the instance of 4 with $\varphi = p$ fails at w_1. Similarly, $\mathfrak{M}, w_2 \nVdash \Diamond\neg p \to \Box\Diamond\neg p$, so the instance of 5 with $\varphi = \neg p$ fails at w_2. □

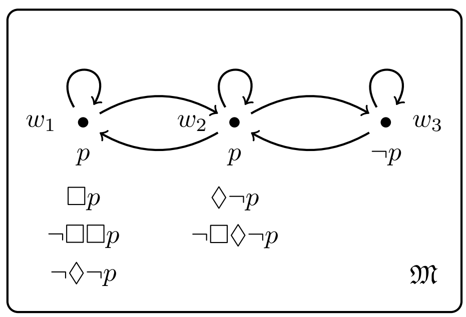


Figure 2.2: The model for Theorem 2.25.

mod:prf:dis: fig:KTBnot45

Theorem 2.26. $\mathbf{KD5} \neq \mathbf{KT4} = \mathbf{S4}$. mod:prf:dis: thm:KD5not4

Proof. By Theorem 1.21 we know that all instances of D and 5 to be true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figure 2.3, and notice that $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$. □

Problem 2.5. Give an alternative proof of Theorem 2.26 using a model with 3 worlds.

Problem 2.6. Provide a single reflexive transitive model showing that both $\mathbf{KT4} \nvdash$ B and $\mathbf{KT4} \nvdash$ 5.

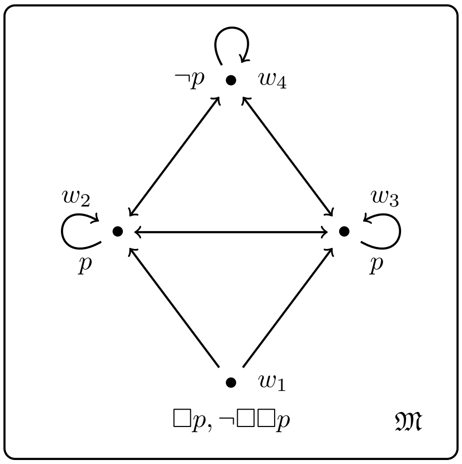


Figure 2.3: The model for Theorem 2.26.

mod:prf:dis: fig:KD5not4

2.10 Derivability from a Set of Formulas

mod:prf:prg: sec

In section 2.7 we defined a notion of provability of a formula in a system Σ. We now extend this notion to provability in Σ from formulas in a set Γ.

mod:prf:prg: defn:Gammaproves

Definition 2.27. A formula φ is derivable in a system Σ from a set of formulas Γ, written $\Gamma \vdash_\Sigma \varphi$ if and only if there are $\psi_1, \dots, \psi_n \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$.

2.11 Properties of Derivability

mod:prf:prp: sec

mod:prf:prp: prop:derivabilityfacts

Proposition 2.28. *Let Σ be a modal system and Γ a set of modal formulas. The following properties hold:*

1. Monotony*: If $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_\Sigma \varphi$;*
2. Reflexivity*: If $\varphi \in \Gamma$ then $\Gamma \vdash_\Sigma \varphi$;*
3. Cut*: If $\Gamma \vdash_\Sigma \varphi$ and $\Delta \cup \{\varphi\} \vdash_\Sigma \psi$ then $\Gamma \cup \Delta \vdash_\Sigma \psi$;*
4. Deduction theorem*: $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi$ if and only if $\Gamma \vdash_\Sigma \psi \to \varphi$;*
5. Rule T*: If $\Gamma \vdash_\Sigma \varphi_1$ and ... and $\Gamma \vdash_\Sigma \varphi_n$ and $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \psi) \cdots)$ is a tautological instance, then $\Gamma \vdash_\Sigma \psi$.* *mod:prf:prp: prop:derivabilityfacts-ruleT*

The proof is an easy exercise. Part (5) of Proposition 2.28 gives us that, for instance, if $\Gamma \vdash_\Sigma \varphi \lor \psi$ and $\Gamma \vdash_\Sigma \neg\varphi$, then $\Gamma \vdash_\Sigma \psi$. Also, in what follows, we write $\Gamma, \varphi \vdash_\Sigma \psi$ instead of $\Gamma \cup \{\varphi\} \vdash_\Sigma \psi$.

Definition 2.29. A set Γ is *deductively closed* relatively to a system Σ if and only if $\Gamma \vdash_\Sigma \varphi$ implies $\varphi \in \Gamma$.

2.12 Consistency

mod:prf:con:sec

Definition 2.30. A set Γ is *consistent* relatively to a system Σ or, as we wil say, Σ-consistent, if and only if $\Gamma \nvdash_\Sigma \bot$.

So for instance, the set $\{\Box(p \to q), \Box p, \neg\Box q\}$ is consistent relatively to propositional logic, but not **K**-consistent. Similarly, the set $\{\Diamond p, \Box\Diamond p \to q, \neg q\}$ is not **K5**-consistent.

Proposition 2.31. *Let Γ be a set of formulas. Then:* *mod:prf:con:prop:consistencyfacts*

1. *A set Γ is Σ-consistent if and only if there is some formula φ such that $\Gamma \nvdash_\Sigma \varphi$.*
2. *$\Gamma \vdash_\Sigma \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is not Σ-consistent.* *mod:prf:con:prop:consistencyfacts-b*
3. *If Γ is Σ-consistent, then for any formula φ, either $\Gamma \cup \{\varphi\}$ is Σ-consistent or $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent.* *mod:prf:con:prop:consistencyfacts-c*

Proof. These fact follow easily using classical propositional logic. We give the argument for (c). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent. Then by (b) both $\Gamma, \varphi \vdash_\Sigma \bot$ and $\Gamma, \neg\varphi \vdash_\Sigma \bot$. By the deduction theorem $\Gamma \vdash_\Sigma \varphi \to \bot$ and $\Gamma \vdash_\Sigma \neg\varphi \to \bot$. But $(\varphi \to \bot) \to ((\neg\varphi \to \bot) \to \bot)$ is a tautological instance, hence by Proposition 2.28(5), $\Gamma \vdash_\Sigma \bot$. $\Box$

Chapter 3

Completeness

3.1 Complete Consistent Sets

mod:com:ccs:
sec

Definition 3.1. A set Γ is *complete Σ-consistent* if and only if it is Σ-consistent and for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

mod:com:ccs:
prop:completeconsproperties

Proposition 3.2. *Suppose Γ is complete Σ-consistent. Then:*

1. *Γ is deductively closed in Σ.*

mod:com:ccs:
prop:completeconsproperties-b

2. *$\Sigma \subseteq \Gamma$.*

mod:com:ccs:
prop:completeconsproperties-c

3. *$\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.*

mod:com:ccs:
prop:completeconsproperties-d

4. *$\varphi \to \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ implies $\psi \in \Gamma$.*

Proof.

1. If $\Gamma \vdash_\Sigma \varphi$ but $\varphi \notin \Gamma$ then by maximality $\neg\varphi \in \Gamma$, and Γ is inconsistent.

2. If $\varphi \in \Sigma$ then $\Gamma \vdash_\Sigma \varphi$, and $\varphi \in \Gamma$ by deductive closure.

3. If $\neg\varphi \in \Gamma$, then by consistency $\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then by maximality $\varphi \in \Gamma$.

4. Suppose $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash_\Sigma \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \to \psi \notin \Gamma$ then by maximality $\neg(\varphi \to \psi) \in \Gamma$, so by Rule T, deductive closure, and consistency both $\varphi \in \Gamma$ and $\psi \notin \Gamma$.

□

3.2 Lindenbaum's Lemma

mod:com:lin:sec

Theorem 3.3 (Lindenbaum's Lemma)**.** *If Γ is Σ-consistent then there is a complete Σ-consistent set Δ extending Γ.* mod:com:lin:thm:lindenbaum

Proof. Let φ_0, φ_1, ... be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing p_0, and at each stage n list the finitely many formulas of length n using only variables among p_0, ..., p_n. We define sets of formulas Δ_n by induction on n, and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$, then supposing that Δ_n has been defined:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

If we now let $\Delta = \bigcup_n \Delta_n$, we can show the following:

1. For each n, $\Delta_n \subseteq \Delta$ (immediate from the definition).
2. $\Gamma \subseteq \Delta$ (from (a)).
3. If $n \leq m$ then $\Delta_n \subseteq \Delta_m$ (by induction on $m - n$).
4. Δ is maximal (by construction).
5. For each m, Δ_m is consistent (by induction on m, using Proposition 2.31(3))
6. If $\Delta' \subseteq \Delta$ is finite, then there is m such that $\Delta' \subseteq \Delta_m$.
7. Δ is consistent.

It follows that Δ is a complete Σ-consistent set extending Γ. □

3.3 Derivability and Complete Consistent Sets

mod:com:pcc:sec

Corollary 3.4. *$\Gamma \vdash_\Sigma \varphi$ if and only if $\varphi \in \Delta$ for each complete Σ-consistent set Δ extending Γ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system Σ.)* mod:com:pcc:cor:provability-characterization

Proof. Suppose $\Gamma \vdash_\Sigma \varphi$, and let Δ be any complete Σ-consistent set extending Γ. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_\Sigma \varphi$ (by monotony) and $\Delta \vdash_\Sigma \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \nvdash_\Sigma \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent, and by Lindenbaum's Lemma there is a complete consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$. By consistency, $\varphi \notin \Delta$. □

3.4 Modalities and Complete Consistent Sets

mod:com:mod:sec When we construct a model whose set of worlds is given by the complete consistent sets in some normal modal logic Σ, we will also need to define an accessibility relation between such "worlds." The next few lemmas give us the tools to do so. As noted, Σ will be a normal modal logic throughout.

mod:com:mod:lem:Gamma-proves1 **Lemma 3.5.** *If $\Gamma \vdash_\Sigma \varphi$ then $\{\Box\psi : \psi \in \Gamma\} \vdash_\Sigma \Box\varphi$.*

Proof. If $\Gamma \vdash_\Sigma \varphi$ then there are $\psi_1, \dots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots(\psi_n \to \varphi)\cdots)$. Since Σ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \to (\Box\psi_2 \to \cdots(\Box\psi_n \to \Box\varphi)\cdots)$, where obviously $\Box\psi_1, \dots, \Box\psi_k \in \{\Box\psi : \psi \in \Gamma\}$. Hence, by definition, $\{\Box\psi : \psi \in \Gamma\} \vdash_\Sigma \Box\varphi$. □

mod:com:mod:lem:Gamma-proves2 **Lemma 3.6.** *If $\{\psi : \Box\psi \in \Gamma\} \vdash_\Sigma \varphi$ then $\Gamma \vdash_\Sigma \Box\varphi$.*

Proof. Let $\Delta = \{\psi : \Box\psi \in \Gamma\}$, so that $\Delta \vdash_\Sigma \varphi$; then by Lemma 3.5, $\{\Box\psi : \psi \in \Delta\} \vdash \Box\varphi$. But obviously $\{\Box\psi : \psi \in \Delta\} \subseteq \Gamma$, so also $\Gamma \vdash_\Sigma \Box\varphi$ by Monotony. □

mod:com:mod:thm:box-phiGamma **Theorem 3.7.** *If Γ is complete Σ-consistent, then $\Box\varphi \in \Gamma$ if and only if for every complete Σ-consistent Δ such that $\{\psi : \Box\psi \in \Gamma\} \subseteq \Delta$, it holds that $\varphi \in \Delta$.*

Proof. The left-to-right half of the theorem is obvious. For the converse, suppose $\Box\varphi \notin \Gamma$. Since Γ is deductively closed, $\Gamma \nvdash_\Sigma \Box\varphi$, and by Lemma 3.6 $\{\psi : \Box\psi \in \Gamma\} \nvdash_\Sigma \varphi$. By Proposition 2.31(2), $\{\psi : \Box\psi \in \Gamma\} \cup \{\neg\varphi\}$ is Σ-consistent, so that by Lindenbaum's Lemma there is a complete Σ-consistent set Δ such that $\{\psi : \Box\psi \in \Gamma\} \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$, and the theorem is proved. □

mod:com:mod:lem:Gamma-proves3 **Lemma 3.8.** *Suppose Γ and Δ are complete Σ-consistent. Then: $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$ if and only if $\{\Diamond\varphi : \varphi \in \Delta\} \subseteq \Gamma$.*

Proof. "Only if" direction: Assume $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$ and suppose $\varphi \in \Delta$. In order to show $\Diamond\varphi \in \Gamma$ it suffices to show $\Box\neg\varphi \notin \Gamma$ for then by maximality $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of Δ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

"If" direction: Assume $\{\Diamond\varphi : \varphi \in \Delta\} \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in Γ by consistency $\Box\varphi \notin \Gamma$, as required. □

Corollary 3.9. *If Γ is complete Σ-consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some complete Σ-consistent Δ such that $\{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma$, it holds that $\varphi \in \Delta$.*

Proof. Suppose Γ is complete Σ-consistent, and argue as follows:

$$\begin{array}{lll}
\Diamond\varphi \in \Gamma & \Leftrightarrow \neg\Box\neg\varphi \in \Gamma, & \text{re-writing;} \\
& \Leftrightarrow \Box\neg\varphi \notin \Gamma, & \Gamma \text{ is complete } \Sigma\text{-con} \\
& \Leftrightarrow \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\psi : \Box\psi \in \Gamma\} \subseteq \Delta \wedge \neg\varphi \notin \Delta], & \text{Theorem 3.7;} \\
& \Leftrightarrow \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma \wedge \neg\varphi \notin \Delta], & \text{Lemma 3.8;} \\
& \Leftrightarrow \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma \wedge \varphi \in \Delta] \Box & \Delta \text{ is complete.}
\end{array}$$

3.5 Canonical Models

mod:com:cmd: sec

Definition 3.10. A model $\mathfrak{M}$ is said to *determine* a normal modal logic Σ precisely when $\mathfrak{M} \models \varphi$ if and only if $\Sigma \vdash \varphi$, for all formulas φ.

Definition 3.11. Let Σ be a normal modal logic. The *canonical model* for Σ is $\mathfrak{M}^\Sigma = \langle W^\Sigma, R^\Sigma, V^\Sigma \rangle$, where:

1. $\mathfrak{M}^\Sigma = \{w \subseteq \mathrm{Frm}(\mathcal{L}) : w \text{ is complete } \Sigma\text{-consistent}\}$.
2. $R^\Sigma ww'$ holds if and only if $\{\varphi : \Box\varphi \in w\} \subseteq w'$.
3. $V^\Sigma(p) = \{w : p \in w\}$.

3.6 The Truth Lemma

mod:com:tru: sec

Proposition 3.12 (Truth Lemma). *For every formula φ, $\mathfrak{M}^\Sigma, w \models \varphi$ if and only if $\varphi \in w$.* *mod:com:tru: prop:truthlemma*

Proof. By induction on φ. *Basis*: if φ is a propositional variable, say p, then:

$$\mathfrak{M}^\Sigma, w \models p \Leftrightarrow w \in V^\Sigma(p) \Leftrightarrow p \in w.$$

If φ is $\bot$ then both $\mathfrak{M}^\Sigma, w \not\models \bot$ and $\bot \notin w$ (by consistency of w). The cases for $\neg\varphi$ and $\varphi \to \psi$ follow from the inductive hypothesis and Proposition 3.2, parts Proposition 3.2 and Proposition 3.2. Here is the case for $\Box\varphi$; in one direction:

$$\begin{array}{lll}
\mathfrak{M}^\Sigma, w \models \Box\varphi & \Rightarrow \forall w' \in W^\Sigma(R^\Sigma ww' \Rightarrow \mathfrak{M}^\Sigma, w' \models \varphi), & \text{def. } \models; \\
& \Rightarrow \forall w' \in W^\Sigma(\{\psi : \Box\psi \in w\} \subseteq w' \Rightarrow \mathfrak{M}^\Sigma, w' \models \varphi), & \text{def. } R^\Sigma; \\
& \Rightarrow \forall w' \in W^\Sigma(\{\psi : \Box\psi \in w\} \subseteq w' \Rightarrow \varphi \in w'), & \text{ind. hyp.;} \\
& \Rightarrow \Box\varphi \in w, & \text{Theorem 3.7.}
\end{array}$$

Conversely, assume $\Box\varphi \in w$, and let w' be an arbitrary world in W^Σ such that $R^\Sigma ww'$. By definition of R^Σ, we have $\{\psi : \Box\psi \in w\} \subseteq w'$, which immediately gives $\varphi \in w'$. By induction hypothesis, $\mathfrak{M}^\Sigma, w' \models \varphi$, and since w' was arbitrary, $\mathfrak{M}^\Sigma, w \models \Box\varphi$. □

3.7 Completeness for K

mod:com:cmk:sec

mod:com:cmk:thm:determination **Theorem 3.13** (Determination). *For every normal modal logic Σ: $\mathfrak{M}^\Sigma \models \varphi$ if and only if $\Sigma \vdash \varphi$.*

Proof. If $\mathfrak{M}^\Sigma \models \varphi$, then for every complete Σ-consistent w, we have $\mathfrak{M}^\Sigma, w \models \varphi$. Hence, by the Truth Lemma, $\varphi \in w$ for every complete Σ-consistent w, whence by Corollary 3.4 (with $\Gamma = \emptyset$), $\Sigma \vdash \varphi$. Conversely, if $\Sigma \vdash \varphi$ then by Proposition 3.2(2), every complete Σ-consistent w contains φ, and hence by the Truth Lemma $\mathfrak{M}^\Sigma, w \models \varphi$ for every w, i.e., $\mathfrak{M}^\Sigma \models \varphi$. □

mod:com:cmk:cor:Kcomplete **Corollary 3.14.** *The basic modal logic* **K** *is complete with respect to the class of all models, i.e., if $\vDash \varphi$ then* $\mathbf{K} \vdash \varphi$.

Proof. Contrapositively, if $\mathbf{K} \nvdash \varphi$ then by Determination $\mathfrak{M}^{\mathbf{K}} \not\models \varphi$ and hence φ is not valid. □

3.8 Frame Completeness

mod:com:fra:sec

The completeness theorem for LogK can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

mod:com:fra:thm:completeframeprops **Theorem 3.15.** *If a normal modal logic Σ contains one of the schemas on the left-hand side of the table of Figure 3.1, then the canonical model for Σ has the corresponding property on the right-hand side.*

If Σ contains ...	*... the canonical model for Σ is:*
D: $\Box\varphi \to \Diamond\varphi$	serial;
T: $\Box\varphi \to \varphi$	reflexive;
B: $\varphi \to \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \to \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \to \Box\Diamond\varphi$	euclidean.

Figure 3.1: Basic correspondence facts.

mod:com:fra:fig:correspondencetable

Proof. We take each of these up in turn. Suppose Σ contains D, and let $w \in W^\Sigma$; we need to show that there is a w' such that $R^\Sigma ww'$. It suffices to show that $\{\psi : \Box\psi \in w\}$ is Σ-consistent, for then by Lindenbaum's Lemma, there is a complete Σ-consistent set $w' \supseteq \{\psi : \Box\psi \in w\}$, and by definition of R^Σ we have $R^\Sigma ww'$. So, suppose for contradiction that $\{\psi : \Box\psi \in w\}$ is *not* Σ-consistent, i.e., $\{\psi : \Box\psi \in w\} \vdash_\Sigma \bot$. By Lemma 3.6, $w \vdash_\Sigma \Box\bot$, and since Σ contains D, also $w \vdash_\Sigma \Diamond\bot$. But Σ is normal, so $\Sigma \vdash \neg\Diamond\bot$ (Proposition 2.6), whence also $w \vdash_\Sigma \neg\Diamond\bot$, against the consistency of w.

Now suppose Σ contains T, and let $w \in W^\Sigma$. We want to show $R^\Sigma ww$, i.e., $\{\varphi : \Box\varphi \in w\} \subseteq w$. But if $\Box\varphi \in w$ then by T also $\varphi \in w$, as desired.

Now suppose Σ contains B, and suppose $R^{\Sigma}uv$ for $u, v \in W^{\Sigma}$. We need to show that $R^{\Sigma}vu$, i.e., $\{\varphi : \Box\varphi \in v\} \subseteq u$. By Lemma 3.8, this is equivalent to $\{\Diamond\varphi : \varphi \in u\} \subseteq v$. So suppose $\varphi \in u$. By B, also $\Box\Diamond\varphi \in u$. By the hypothesis that $R^{\Sigma}uv$, we have that $\{\psi : \Box\psi \in u\} \subseteq v$, and hence $\Diamond\varphi \in v$, as required.

Now suppose Σ contains 4, and suppose $R^{\Sigma}uv$ and $R^{\Sigma}vw$. We need to show $R^{\Sigma}uw$. From the hypothesis we have both $\{\psi : \Box\psi \in u\} \subseteq v$ and $\{\psi : \Box\psi \in v\} \subseteq w$. In order to show $R^{\Sigma}uw$ it suffices to show $\{\psi : \Box\psi \in u\} \subseteq w$. So let $\Box\psi \in u$; by 4, also $\Box\Box\psi \in u$ and by hypothesis we get, first, that $\Box\psi \in v$ and, second, that $\psi \in w$, as desired.

Now suppose Σ contains 5, suppose $R^{\Sigma}uv$ and $R^{\Sigma}uw$. We need to show $R^{\Sigma}vw$. The first hypothesis give $\{\varphi : \Box\varphi \in u\} \subseteq v$, and the second hypothesis is equivalent to $\{\Diamond\varphi : \varphi \in w\} \subseteq u$, by Lemma 3.8. To show $R^{\Sigma}vw$, by Lemma 3.8, it suffices to show $\{\Diamond\varphi : \varphi \in w\} \subseteq v$. So let $\varphi \in w$; by the second hypothesis $\Diamond\varphi \in u$ and by 5, $\Box\Diamond\varphi \in u$ as well. But now the first hypothesis give $\Diamond\varphi \in v$, as desired. □

As a corollary we obtain completeness results for a number of systems. For instance, we know that **S5** = **KT5** = **KTB4** is complete with respect to the class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

Theorem 3.16. *Let $\mathcal{C}_{\mathrm{D}}$, $\mathcal{C}_{\mathrm{T}}$, $\mathcal{C}_{\mathrm{B}}$, $\mathcal{C}_4$, and $\mathcal{C}_5$ be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1, \ldots, \varphi_n$ among* D, T, B, 4, *and* 5, *the system $\mathbf{K}\varphi_1 \ldots \varphi_n$ is determined by the class of models $\mathcal{C} = \mathcal{C}_{\varphi_1} \cap \cdots \cap \mathcal{C}_{\varphi_n}$.* *mod:com:fra: thm:generaldet*

Proposition 3.17. *Let Σ be a normal modal logic; then:* *mod:com:fra: prop:anotherfive-a*

1. *If Σ contains the schema $\Diamond\varphi \to \Box\varphi$ then the canonical model for Σ is partially functional.*

2. *If Σ contains the schema $\Diamond\varphi \leftrightarrow \Box\varphi$ then the canonical model for Σ is functional.*

3. *If Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ then the canonical model for Σ is weakly dense.*

(see Figure 1.4 for definitions of these frame properties).

Proof. 1. suppose that Σ contains the schema $\Diamond\varphi \to \Box\varphi$, to show that R^{Σ} is partialy functional we need to prove that for any $u, v, w \in W^{\Sigma}$, if $R^{\Sigma}wu$ and $R^{\Sigma}wv$ then $u = v$. Since $R^{\Sigma}wu$ we have $\{\varphi : \Box\varphi \in w\} \subseteq u$ and since $R^{\Sigma}wv$ also $\{\varphi : \Box\varphi \in w\} \subseteq v$. The identity $u = v$ will follow if we can establish the two inclusions $u \subseteq v$ and $v \subseteq u$. For the first inclusion, let $\varphi \in u$; then $\Diamond\varphi \in w$, and by the schema and deductive closure of w also $\Box\varphi \in w$, whence by the hypothesis that $R^{\Sigma}wv$, $\varphi \in v$. The second inclusion is similar, so this establishes part (a).

2. This follows immediately from part (1) and the seriality proof in Theorem 3.15.

3. Suppose Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ and to show that R^{Σ} is weakly dense, let $R^{\Sigma}uv$. We need to show that there is a complete Σ-consistent set w such that $R^{\Sigma}uw$ and $R^{\Sigma}wv$. Let:

$$\Gamma = \{\varphi : \Box\varphi \in u\} \cup \{\Diamond\psi : \psi \in v\}.$$

It suffices to show that Γ is Σ-consistent, for then by Lindenbaum's Lemma it can be extendend to a complete Σ-consistent set w such that $\{\varphi : \Box\varphi \in u\} \subseteq w$ and $\{\Diamond\psi : \psi \in v\} \subseteq w$, i.e., $R^{\Sigma}uw$ and $R^{\Sigma}wv$.

Suppose for contradiction that Γ is not consistent. Then there are formulas $\Box\varphi_1, \dots, \Box\varphi_n \in u$ and $\psi_1, \dots, \psi_m \in v$ such that $\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_{\Sigma} \bot$. Since $\Diamond(\psi_1 \land \dots \land \psi_m) \to (\Diamond\psi_1 \land \dots \land \Diamond\psi_m)$ is derivable in every normal modal logic, we argue as follows, contradicting the consistency of v:

$$\begin{array}{llll}
\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_{\Sigma} \bot & \Rightarrow & \varphi_1, \dots, \varphi_n \vdash_{\Sigma} (\Diamond\psi_1 \land \dots \land \Diamond\psi_m) \to \bot, & \text{deduction theorem;} \\
 & \Rightarrow & \varphi_1, \dots, \varphi_n \vdash_{\Sigma} \Diamond(\psi_1 \land \dots \land \psi_m) \to \bot, & \Sigma \text{ is normal;} \\
 & \Rightarrow & \varphi_1, \dots, \varphi_n \vdash_{\Sigma} \Box\neg(\psi_1 \land \dots \land \psi_m), & \text{PL, re-writing;} \\
 & \Rightarrow & \Box\varphi_1, \dots, \Box\varphi_n \vdash_{\Sigma} \Box\Box\neg(\psi_1 \land \dots \land \psi_m), & \text{Lemma 3.5;} \\
 & \Rightarrow & \Box\varphi_1, \dots, \Box\varphi_n \vdash_{\Sigma} \Box\neg(\psi_1 \land \dots \land \psi_m), & \text{by the schema;} \\
 & \Rightarrow & u \vdash_{\Sigma} \Box\neg(\psi_1 \land \dots \land \psi_m), & \text{Monotony;} \\
 & \Rightarrow & \Box\neg(\psi_1 \land \dots \land \psi_m) \in u, & \text{deductive closure;} \\
 & \Rightarrow & \neg(\psi_1 \land \dots \land \psi_m) \in v, \qquad \Box & \text{since } R^{\Sigma}uv.
\end{array}$$

On the strength of these examples, one might think that every system Σ of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of Σ is valid. Unfortunately, there are many systems that are not complete in this sense.

Chapter 4

Filtrations and Decidability

4.1 Preliminaries

mod:fil:pre: sec

Filtrations allow us to establish the decidability of our systems of modal logic by showing that they have the *finite model property*, i.e., that any formula that is true (false) in a model is also true (false) in a *finite* model.

mod:fil:pre: def:modallyclosed

Definition 4.1. A set Γ of formulas is *closed under subformulas* if it contains every subformula of a formula in Γ. Further, Γ is *modally closed* if it is closed under subformulas and moreover $\varphi \in \Gamma$ implies $\Box\varphi, \Diamond\varphi \in \Gamma$.

Definition 4.2. Let $\mathfrak{M} = \langle W, R, V \rangle$ and suppose Γ is closed under subformulas. Define a relation $\equiv$ on W to hold of any two worlds that make true the same formulas from Γ, i.e.:

$$u \equiv v \quad \text{if and only if} \quad \forall \varphi \in \Gamma : \mathfrak{M}, u \models \varphi \Leftrightarrow \mathfrak{N}, v \models \varphi.$$

Clearly, $\equiv$ is an equivalence relation over W. Standardly, for any $w \in W$, the equivalence class of w is denoted by $[w]$.

4.2 Filtrations

mod:fil:fil: sec

mod:fil:fil: def:filtration

Definition 4.3. Let Γ be closed under subformulas and $\mathfrak{M} = \langle W, R, V \rangle$. A *filtration of* $\mathfrak{M}$ *through* Γ is any model $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$, where:

1. $W^* = \{[w] : w \in W\}$;
2. For any $u, v \in W$: (mod:fil:fil: def:filtration-R)
   a) If Ruv then $R^*[u][v]$; (mod:fil:fil: def:filtration-R1)
   b) If $R^*[u][v]$ then for any $\Box\varphi \in \Gamma$, if $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$; (mod:fil:fil: def:filtration-R2)
   c) If $R^*[u][v]$ then for any $\Diamond\varphi \in \Gamma$, if $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. (mod:fil:fil: def:filtration-R3)

3. $V^*(p) = \{[u] : u \in V(p)\}$.

Theorem 4.4. *If $\mathfrak{M}^*$ is a filtration of $\mathfrak{M}$ through Γ, then for every $\varphi \in \Gamma$ and $w \in W$, we have $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M}^*, [w] \models \varphi$.* mod:fil:fil:thm:filtrations

Proof. By induction on φ, using the fact that Γ is closed under subformulas. For φ atomic, say p: the left-to-right direction is immediate, as $\mathfrak{M}, w \models p$ only if $w \in V(p)$, which implies $[w] \in V^*(p)$, i.e., $\mathfrak{M}^*, [w] \models p$. Conversely, suppose $\mathfrak{M}^*, [w] \models p$, i.e., $[w] \in V^*(p)$; then $w \equiv w' \in V(p)$, and since $p \in \Gamma$, also $w \in V(p)$, so that $\mathfrak{M}, w \models p$. The cases for the Boolean connectives follow immediately from the inductive hypothesis and closure of Γ under subformulas.

So we do the case for $\Box\varphi \in \Gamma$. Suppose $\mathfrak{M}, u \models \Box\varphi$; to show that $\mathfrak{M}^*, [u] \models \varphi$, let v be such that $R^*[u][v]$. From Definition 4.3(2b), we have that $\mathfrak{M}, v \models \varphi$, and by inductive hypothesis $\mathfrak{M}^*, [v] \models \varphi$. Since v was arbitrary, $\mathfrak{M}^*, [u] \models \Box\varphi$ follows. Conversely, suppose $\mathfrak{M}^*, [u] \models \Box\varphi$ and let v be arbitrary such that Ruv. From Definition 4.3(2a), we have $R^*[u][v]$, so that $\mathfrak{M}^*, [v] \models \varphi$; by inductive hypothesis $\mathfrak{M}, v \models \varphi$, and since v was arbitrary, $\mathfrak{M}, u \models \Box\varphi$. □

Corollary 4.5. *Let Γ be closed under subformulas. Then:*

1. *If $\mathfrak{M}^*$ is a filtration of $\mathfrak{M}$ through Γ then for any $\varphi \in \Gamma$: $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{M}^* \models \varphi$.*

2. *If $\mathcal{C}$ is a class of models and $\Gamma(\mathcal{C})$ is the class of Γ-filtrations of models in $\mathcal{C}$, then any formula $\varphi \in \Gamma$ is valid in $\mathcal{C}$ if and only if it is valid in $\Gamma(\mathcal{C})$.*

4.3 Examples of Filtrations

We have not yet shown that there are any filtrations. But indeed, for any model $\mathfrak{M}$, there are many filtrations of $\mathfrak{M}$ through Γ. We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as Definition 4.3 stipulates directly what W^* and V^* should be like). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible. mod:fil:exf:sec

Definition 4.6. Where Γ is closed under subformulas, the *finest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting:

$$R^*[u][v] \quad \text{if and only if} \quad \exists u' \in [u]\, \exists v' \in [v] : Ru'v'.$$

Proposition 4.7. *The finest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Proof. We need to check that R^*, so defined, satisfies Definition 4.3(2). We check the three conditions in turn.

If Ruv then by reflexivity of $\equiv$, also $R^*[u][v]$, so (2a) is satisfied.

For (2b), suppose $\Box\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, u \models \Box\varphi$. By definition of R^*, there are $u' \equiv u$ an $v' \equiv v$ such that $Ru'v'$. Since u and u' agree on Γ, also

$\mathfrak{M}, u' \models \Box\varphi$, so that $\mathfrak{M}, v' \models \varphi$. By closure of Γ, v and v' agree on φ, so $\mathfrak{M}, v \models \varphi$, as desired.

To verify (2c), suppose $\Diamond\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, v \models \varphi$. Arguing similarly to the previous case, $\mathfrak{M}, u \models \Diamond\varphi$. $\square$

Definition 4.8. Where Γ is closed under subformulas, the *coarsest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting $R^*[u][v]$ if and only if *both* of the following conditions are met:

1. If $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$;
2. If $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$.

Proposition 4.9. *The coarsest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Proof. Given the definition of R^*, the only condition that is left to verify is the implication from Ruv to $R^*[u][v]$. Assuming Ruv, suppose $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$; then obviously $\mathfrak{M}, v \models \varphi$. Similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. $\square$

4.4 Filtrations are Finite

mod:fil:fin:sec

mod:fil:fin:prop:filt-are-finite **Proposition 4.10.** *If Γ is finite then any filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ through Γ is also finite.*

Proof. If $u \equiv v$ then, by Theorem 4.4, the set of $\varphi \in \Gamma$ that are true at u is the same as the set of $\varphi \in \Gamma$ that are true at v. So to each $[u] \in W^*$ we can assign a *distinct* subset of Γ. Hence if Γ contains n sentences the cardinality of W^* is no greater than 2^n. $\square$

4.5 S5 has the Finite Model Property

mod:fil:fmp:sec

Definition 4.11. A system Σ of modal logic is said to have the *finite model property* if whenever a formula φ is true at a world in a model of Σ then φ is true at a world in a *finite* model of Σ.

mod:fil:fmp:prop:univ-fin **Proposition 4.12.** *Let $\mathcal{U}$ be the class of universal models (see Proposition 1.29) and $\mathcal{U}_{\mathrm{Fin}}$ the class of all finite universal models. Then any formula φ is valid in $\mathcal{U}$ if and only if it is valid in $\mathcal{U}_{\mathrm{Fin}}$.*

Proof. Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to left direction, suppose that φ is false at some world w in a universal model $\mathfrak{M}$. Let Γ contain φ as well as all of its subformulas; clearly Γ is finite. Take a filtration $\mathfrak{M}^*$ of $\mathfrak{M}$; then $\mathfrak{M}^*$ is finite by

Proposition 4.10, and by Theorem 4.4, φ is false at $[w]$ in $\mathfrak{M}^*$. It remains to observe that $\mathfrak{M}^*$ is also universal: given u and v, by hypothesis Ruv and by Definition Definition 4.3(2), also $R^*[u][v]$. □

Corollary 4.13. **S5** *has the finite model property.* *mod:fil:fmp: cor:S5fmp*

Proof. By Proposition 1.29 and Proposition 4.12, if φ is true at a world in some reflexive and euclidean model then it is true at a world in a finite universal model (universal models are obviously reflexive and euclidean). □

Problem 4.1. Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

Problem 4.2. Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

4.6 S5 is Decidable

The finite model property gives us an easy way to show that systems of modal logic given by schemas are *decidable* (i.e., that there is a computable procedure to determine whether a formulas is derivable in the system or not). mod:fil:dec: sec

Theorem 4.14. **S5** *is decidable.*

Proof. Let φ be given, and suppose the propositional variables occurring in φ are among $p_1, \ldots, p_k$. Since for each n there are only finitely many models with n worlds assigning a value to $p_1, \ldots, p_k$, we can enumerate, *in parallel*, all the theorems of **S5** by generating proofs in some systematic way; and all the models containing 1, 2, ... worlds and checking whether φ fails at a world in some such model. Eventually one of the two parallel processes will give an answer, as by Theorem 3.16 and Corollary 4.13, either φ is derivable or it fails in a finite universal model. □

The above proof works for **S5** because filtrations of universal models are automatically universal. The same holds for reflexivity and seriality, but more work is needed for other properties.

Problem 4.3. Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

Problem 4.4. Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

4.7 Filtrations and Properties of Accessibility

mod:fil:acc:sec

Definition 4.15. Let Γ be closed under subformulas and $\mathfrak{M} = \langle W, R, V\rangle$ a model. Then we can define conditions on pairs of worlds u, v as given in the table of Figure 4.1.

$C_1(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$;
$C_2(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then $\mathfrak{M}, u \models \varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$;
$C_3(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Diamond\varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$;
$C_4(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then $\mathfrak{M}, u \models \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Diamond\varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$;

Figure 4.1: Conditions on possible worlds for defining filtrations.

mod:fil:acc:fig:Cn-filtrations
mod:fil:acc:thm:more-filtrations

Theorem 4.16. *Let $\mathfrak{M} = \langle W, R, P\rangle$ be a model, Γ closed under subformulas. Let W^* and V^* be defined as in Definition 4.3. Then:*

1. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_2(u,v)$ then R^* is symmetric, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ is a filtration if $\mathfrak{M}$ is symmetric.*

2. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_3(u,v)$ then R^* is transitive, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ is a filtration if $\mathfrak{M}$ is transitive.*

3. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_2(u,v) \wedge C_3(u,v) \wedge C_4(u,v)$ then R^* is symmetric and transitive, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ is a filtration if $\mathfrak{M}$ is symmetric and transitive.*

4. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_3(u,v) \wedge C_4(u,v)$ then R^* is transitive and euclidean, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ is a filtration if $\mathfrak{M}$ is transitive and euclidean.*

Proof. 1. It's immediate that R^* is symmetric, since $C_1(u,v) \Leftrightarrow C_2(v,u)$ and $C_2(u,v) \Leftrightarrow C_1(v,u)$. So it's left to show that if $\mathfrak{M}$ is symmetric then $\mathfrak{M}^*$ is a filtration through Γ. By condition $C_1(u,v)$ we get that: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$, and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. So all we need is that Ruv implies $R^*[u][v]$.

So suppose Ruv, to show $R^*[u][v]$ we need $C_1(u,v) \wedge C_2(u,v)$. For C_1: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then also $\mathfrak{M}, v \models \varphi$ (since Ruv); and similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. For C_2: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then Ruv implies Rvu by symmetry, so that $\mathfrak{M}, u \models \varphi$; similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$ (since Rvu by symmetry).

2. Exercise.

3. Exercise.

4. Exercise.

□

Problem 4.5. Complete the proof of Theorem 4.16.

4.8 Filtrations of Euclidean Models

The approach of section 4.7 does not work in the case of models that are euclidean or serial and euclidean. Consider the model at the top of Figure 4.2, which is both euclidean and serial. Let $\Gamma = \{p, \Box p\}$. When taking a filtration through Γ, then $[w_1] = [w_3]$ since w_1 and w_3 are the only worlds that agree on Γ. Any filtration will also have the arrow inherited from $\mathfrak{M}$, as depicted in Figure 4.3. But we cannot add arrows to that model in order to make it euclidean, for then there would be a double arrow between w_2 and w_4, and hence also between w_2 and w_5. But $\Box p$ is true at w_2 while p is false at w_5. mod:fil:euc: sec

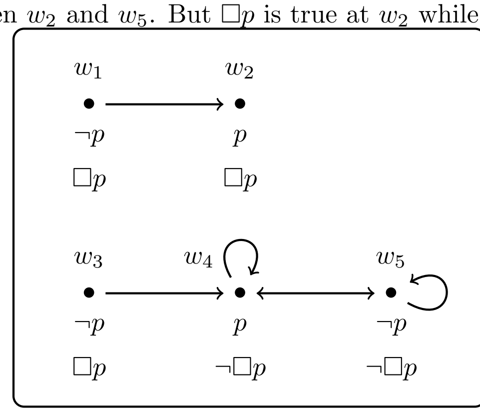


Figure 4.2: A serial and euclidean model.

mod:fil:euc: fig:ser-eucl

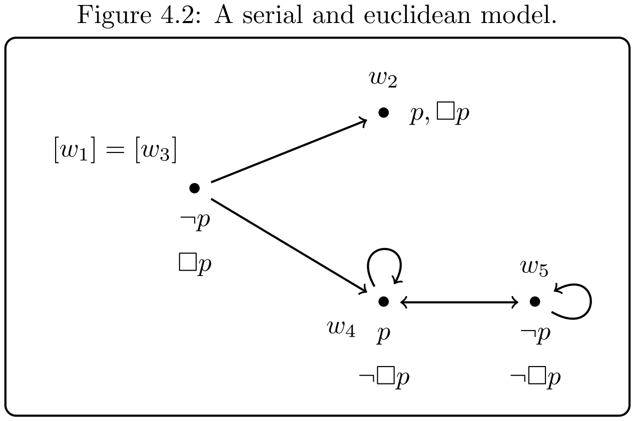


Figure 4.3: The filtration of the model in Figure 4.2.

mod:fil:euc: fig:ser-eucl2

In particular, it is not enough to consider filtrations through arbitrary Γ's closed under subsentences. Instead we need to consider sets Γ that are *modally closed* (see Definition 4.1). Such sets of sentences are infinite, and therefore do not lead immediately to the decidability of the corresponding system.

Theorem 4.17. *Let Γ be modally closed and $\mathfrak{M} = \langle W, R, V \rangle$. If $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a coarsest filtration of $\mathfrak{M}$, then $\mathfrak{M}^*$ is symmetric, transitive or euclidean if $\mathfrak{M}$ is symmetric, transitive, or euclidean, respectively.*

Proof. The proof of transitivity uses the validity of both 4 and $4_\Diamond$ in all transitive models, and likewise euclideanness uses the fact that both 5 and $5_\Diamond$ are valid in all euclidean models, and the proof of symmetry likewise uses both B and $\mathrm{B}_\Diamond$.

If $\mathfrak{M}^*$ is a coarsest filtration, then by definition $R^*[u][v]$ holds if and only if $C_1(u, v)$. For transitivity, suppose $C_1(u, v)$ and $C_1(v, w)$: to show $C_1(u, w)$ suppose $\mathfrak{M}, u \models \Box\varphi$; then $\mathfrak{M}, u \models \Box\Box\varphi$; since $\Box\Box\varphi \in \Gamma$ by closure, also by $C_1(u, v)$, $\mathfrak{M}, v \models \Box\varphi$ and by $C_1(v, w)$, also $\mathfrak{M}, w \models \varphi$. The case for $\Diamond\varphi$ is similar. $\Box$

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Bibliography