frd.1 Second-order Definability

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Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal formula is true at a world are related to first-order formulas. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place predicate symbol Q for the accessibility relation, but also a one-place predicate symbol P_i for the propositional variables p_i occurring in φ.

Definition frd.1. The *standard translation* $\mathrm{ST}_x(\varphi)$ is inductively defined as follows:

1. $\varphi \equiv \bot$: $\mathrm{ST}_x(\varphi) = \bot$.
2. $\varphi \equiv \bot$: $\mathrm{ST}_x(\varphi) = \top$.
3. $\varphi \equiv p_i$: $\mathrm{ST}_x(\varphi) = P_i(x)$.
4. $\varphi \equiv \neg\psi$: $\mathrm{ST}_x(\varphi) = \neg\mathrm{ST}_x(\psi)$.
5. $\varphi \equiv (\psi \land \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \land \mathrm{ST}_x(\chi))$.
6. $\varphi \equiv (\psi \lor \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \lor \mathrm{ST}_x(\chi))$.
7. $\varphi \equiv (\psi \to \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \to \mathrm{ST}_x(\chi))$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \leftrightarrow \mathrm{ST}_x(\chi))$.
9. $\varphi \equiv \Box\psi$: $\mathrm{ST}_x(\varphi) = \forall y\,(Q(x,y) \to \mathrm{ST}_y(\psi))$.
10. $\varphi \equiv \Diamond\psi$: $\mathrm{ST}_x(\varphi) = \exists y\,(Q(x,y) \land \mathrm{ST}_y(\psi))$.

For instance, $\mathrm{ST}_x(\Box p \to p)$ is $(\forall y\,(Q(x,y) \to P(x)) \to P(x)$. Any structure for the language of $\mathrm{ST}_x(\varphi)$ requires a domain, a two-place relation assigned to Q, and subsets of the domain assigned to the one-place predicate symbols P_i. In other words, the components of such a structure are exactly those of a model for φ: the domain is the set of worlds, the two-place relation assigned to Q is the accessibility relation, and the subsets assigned to P_i are just the assignments $V(p_i)$. It won't surprise that satisfaction of φ in a modal model and of $\mathrm{ST}_x(\varphi)$ in the corresponding structure agree:

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Proposition frd.2. *Let $\mathfrak{M} = \langle W, R, V\rangle$, $\mathfrak{M}'$ be the first-order structure with $|\mathfrak{M}'| = W$, $Q^{\mathfrak{M}} = R$, and $P_i^{\mathfrak{M}'} = V(p_i)$, and $s(x) = w$. Then*

$$\mathfrak{M}, w \Vdash \varphi \textit{ iff } \mathfrak{M}', s \models \mathrm{ST}_x(\varphi)$$

Proof. By induction on φ. □

Proposition frd.3. *Suppose φ is a modal formula and $\mathfrak{F} = \langle W, R\rangle$ is a frame. Let $\mathfrak{F}'$ be the first-order structure with $|\mathfrak{F}'| = W$ and $Q^{\mathfrak{F}'} = R$, and let φ' be the second-order formula*

$$\forall X_1 \ldots \forall X_n \,\forall x\, \mathrm{ST}_x(\varphi)[X_1/P_1, \ldots, X_n/P_n],$$

where P_1, …, P_n are all one-place predicate symbols in $\mathrm{ST}_x(\varphi)$. Then

$$\mathfrak{F} \Vdash \varphi \text{ iff } \mathfrak{F}' \models \varphi'$$

Proof. $\mathfrak{F}' \models \varphi'$ iff for every structure $\mathfrak{M}'$ where $P_i^{\mathfrak{M}'} \subseteq W$ for $i = 1$, …, n, and for every s with $s(x) \in W$, $\mathfrak{M}', s \models \mathrm{ST}_x(\varphi)$. By Proposition frd.2, that is the case iff for all models $\mathfrak{M}$ based on $\mathfrak{F}$ and every world $w \in W$, $\mathfrak{M}, w \Vdash \varphi$, i.e., $\mathfrak{F} \Vdash \varphi$. □

Definition frd.4. A class $\mathcal{C}$ of frames is *second-order definable* if there is a sentence φ in the second-order language with a single two-place predicate symbol P and quantifiers only over monadic set variables such that $\mathfrak{F} = \langle W, R\rangle \in \mathcal{C}$ iff $\mathfrak{M} \models \varphi$ in the structure $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $P^{\mathfrak{M}} = R$.

Corollary frd.5. *If a class of frames is definable by a formula φ, the corresponding class of accessibility relations is definable by a monadic second-order sentence.*

Proof. The monadic second-order sentence φ' of the preceding proof has the required property. □

As an example, consider again the formula $\Box p \to p$. It defines reflexivity. Reflexivity is of course first-order definable by the sentence $Q(x, x)$. But it is also definable by the monadic second-order sentence

$$\forall X \,\forall x\, (\forall y\, (Q(x, y) \to X(y)) \to X(x)).$$

This means, of course, that the two sentences are equivalent. Here's how you might convince yourself of this directly: First suppose the second-order sentence is true in a structure M. Since x and X is universally quantified, the remainder must hold for any $x \in W$ and set $X \subseteq W$, e.g, the set $\{z : Rxz\}$ where $R = Q^{\mathfrak{M}}$. So, for any s with $s(x) \in W$ and $s(X) = \{z : Rzx\}$ we have $\mathfrak{M} \models \forall y\, (Q(x, y) \to X(y)) \to X(x)$. But by the way we've picked $s(X)$ that means $\mathfrak{M}, s \models \forall y\, (Q(x, y) \to Q(x, y)) \to Q(x, x)$, which is equivalent to $Q(x, x)$ since the antecedent is valid. Since $s(x)$ is arbitrary, we have $\mathfrak{M} \models \forall x\, Q(x, x)$.

Now suppose that $\mathfrak{M} \models Q(x, x)$ and show that $\mathfrak{M} \models \forall X \,\forall x\, (\forall y\, (Q(x, y) \to X(y)) \to X(x))$. Pick any assignment s, and assume $\mathfrak{M}, s \models \forall y\, (Q(x, y) \to X(y))$. Let s' be the y-variant of s with $s'(y) = x$; we have $\mathfrak{M}, s' \models Q(x, y) \to X(y))$, i.e., $\mathfrak{M}, s \models Q(x, x) \to X(x))$. Since $\mathfrak{M} \models \forall x\, Q(x, x)$, the antecedent is true, and we have $\mathfrak{M}, s \models X(x)$, which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic-second order sentence of the form φ' is equivalent to a first-order sentence. There is no effective method to decide which ones are.

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Bibliography