frd.1  Second-order Definability

Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal formula is true at a world are related to first-order formulas. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place predicate symbol \( Q \) for the accessibility relation, but also a one-place predicate symbol \( P_i \) for the propositional variables \( p_i \) occurring in \( \varphi \).

Definition frd.1. The standard translation \( \text{ST}_x(\varphi) \) is inductively defined as follows:

1. \( \varphi \equiv \bot \): \( \text{ST}_x(\varphi) = \bot \).
2. \( \varphi \equiv \top \): \( \text{ST}_x(\varphi) = \top \).
3. \( \varphi \equiv p_i \): \( \text{ST}_x(\varphi) = P_i(x) \).
4. \( \varphi \equiv \neg \psi \): \( \text{ST}_x(\varphi) = \neg \text{ST}_x(\psi) \).
5. \( \varphi \equiv (\psi \land \chi) \): \( \text{ST}_x(\varphi) = (\text{ST}_x(\psi) \land \text{ST}_x(\chi)) \).
6. \( \varphi \equiv (\psi \lor \chi) \): \( \text{ST}_x(\varphi) = (\text{ST}_x(\psi) \lor \text{ST}_x(\chi)) \).
7. \( \varphi \equiv (\psi \rightarrow \chi) \): \( \text{ST}_x(\varphi) = (\text{ST}_x(\psi) \rightarrow \text{ST}_x(\chi)) \).
8. \( \varphi \equiv (\psi \leftrightarrow \chi) \): \( \text{ST}_x(\varphi) = (\text{ST}_x(\psi) \leftrightarrow \text{ST}_x(\chi)) \).
9. \( \varphi \equiv \Box \psi \): \( \text{ST}_x(\varphi) = \forall y (Q(x, y) \rightarrow \text{ST}_y(\psi)) \).
10. \( \varphi \equiv \Diamond \psi \): \( \text{ST}_x(\varphi) = \exists y (Q(x, y) \land \text{ST}_y(\psi)) \).

For instance, \( \text{ST}_x(\Box p \rightarrow p) \) is \( \forall y (Q(x, y) \rightarrow P(y)) \rightarrow P(x) \). Any structure for the language of \( \text{ST}_x(\varphi) \) requires a domain, a two-place relation assigned to \( Q \), and subsets of the domain assigned to the one-place predicate symbols \( P_i \). In other words, the components of such a structure are exactly those of a model for \( \varphi \): the domain is the set of worlds, the two-place relation assigned to \( Q \) is the accessibility relation, and the subsets assigned to \( P_i \) are just the assignments \( V(p_i) \). It won’t surprise that satisfaction of \( \varphi \) in a modal model and of \( \text{ST}_x(\varphi) \) in the corresponding structure agree:

Proposition frd.2. Let \( \mathcal{M} = \langle W, R, V \rangle \), \( \mathcal{M}' \) be the first-order structure with \( |\mathcal{M}'| = W \), \( Q^{\mathcal{M}'} = R \), and \( P_i^{\mathcal{M}'} = V(p_i) \), and \( s(x) = w \). Then

\[ \mathcal{M}, w \models \varphi \iff \mathcal{M}', s \models \text{ST}_x(\varphi) \]

Proof. By induction on \( \varphi \).  

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Proposition frd.3. Suppose \( \varphi \) is a modal formula and \( \mathfrak{F} = \langle W, R \rangle \) is a frame. Let \( \mathfrak{F}' = \langle W', R' \rangle \) be the first-order structure with \( |\mathfrak{F}'| = W \) and \( Q^{\mathfrak{F}'} = R' \), and let \( \varphi' \) be the second-order formula
\[
\forall X_1 \ldots \forall X_n \forall x \text{ST}_x(\varphi)[X_1/P_1, \ldots, X_n/P_n],
\]
where \( P_1, \ldots, P_n \) are all one-place predicate symbols in \( \text{ST}_x(\varphi) \). Then
\[
\mathfrak{F} \models \varphi \iff \mathfrak{F}' \models \varphi'.
\]
Proof. \( \mathfrak{F} \models \varphi \iff \mathfrak{F}' \models \varphi' \)

Definition frd.4. A class \( \mathcal{F} \) of frames is second-order definable if there is a sentence \( \varphi \) in the second-order language with a single two-place predicate symbol \( P \) and quantifiers only over monadic set variables such that \( \mathfrak{F} = \langle W, R \rangle \in \mathcal{F} \iff \mathfrak{M} \models \varphi \) in the structure \( \mathfrak{M} \) with \( |\mathfrak{M}| = W \) and \( P^{\mathfrak{M}} = R \).

Corollary frd.5. If a class of frames is definable by a formula \( \varphi \), the corresponding class of accessibility relations is definable by a monadic second-order sentence.

Proof. The monadic second-order sentence \( \varphi' \) of the preceding proof has the required property. \( \square \)

As an example, consider again the formula \( \Box p \rightarrow p \). It defines reflexivity. Reflexivity is of course first-order definable by the sentence \( \forall x Q(x, x) \). But it is also definable by the monadic second-order sentence
\[
\forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)).
\]
This means, of course, that the two sentences are equivalent. Here’s how you might convince yourself of this directly: First suppose the second-order sentence is true in a structure \( \mathfrak{M} \). Since \( x \) and \( X \) are universally quantified, the remainder must hold for any \( x \in W \) and set \( X \subseteq W \), e.g., the set \( \{ z : Rxz \} \) where \( R = Q^{\mathfrak{M}} \). So, for any \( s \) with \( s(x) \in W \) and \( s(X) = \{ z : Rxz \} \) we have \( \mathfrak{M} \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x) \). But by the way we’ve picked \( s(X) \) that means \( \mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow Q(x, x) \), which is equivalent to \( Q(x, x) \) since the antecedent is valid. Since \( s(x) \) is arbitrary, we have \( \mathfrak{M} \models \forall x Q(x, x) \).

Now suppose that \( \mathfrak{M} \models \forall x Q(x, x) \) and show that \( \mathfrak{M} \models \forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)) \). Pick any assignment \( s \), and assume \( \mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y)) \). Let \( s' \) be the \( y \)-variant of \( s \) with \( s'(y) = s(x) \); we have \( \mathfrak{M}, s' \models Q(x, y) \rightarrow X(y) \), i.e., \( \mathfrak{M}, s \models Q(x, x) \rightarrow X(x) \). Since \( \mathfrak{M} \models \forall x Q(x, x) \), the antecedent is true, and we have \( \mathfrak{M}, s \models X(x) \), which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic second-order sentence of the form \( \varphi' \) is equivalent to a first-order sentence. There is no effective method to decide which ones are.
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Bibliography