## frd.1 Second-order Definability

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Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal formula is true at a world are related to first-order formulas. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place predicate symbol $Q$ for the accessibility relation, but also a one-place predicate symbol $P_i$ for the propositional variables $p_i$ occurring in $\varphi$.

**Definition frd.1.** The *standard translation* $\mathrm{ST}_x(\varphi)$ is inductively defined as follows:

1. $\varphi \equiv \bot$: $\mathrm{ST}_x(\varphi) = \bot$.
2. $\varphi \equiv \bot$: $\mathrm{ST}_x(\varphi) = \top$.
3. $\varphi \equiv p_i$: $\mathrm{ST}_x(\varphi) = P_i(x)$.
4. $\varphi \equiv \neg\psi$: $\mathrm{ST}_x(\varphi) = \neg\mathrm{ST}_x(\psi)$.
5. $\varphi \equiv (\psi \land \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \land \mathrm{ST}_x(\chi))$.
6. $\varphi \equiv (\psi \lor \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \lor \mathrm{ST}_x(\chi))$.
7. $\varphi \equiv (\psi \to \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \to \mathrm{ST}_x(\chi))$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \leftrightarrow \mathrm{ST}_x(\chi))$.
9. $\varphi \equiv \Box\psi$: $\mathrm{ST}_x(\varphi) = \forall y\,(Q(x,y) \to \mathrm{ST}_y(\psi))$.
10. $\varphi \equiv \Diamond\psi$: $\mathrm{ST}_x(\varphi) = \exists y\,(Q(x,y) \land \mathrm{ST}_y(\psi))$.

For instance, $\mathrm{ST}_x(\Box p \to p)$ is $(\forall y\,(Q(x,y) \to P(x)) \to P(x)$. Any structure for the language of $\mathrm{ST}_x(\varphi)$ requires a domain, a two-place relation assigned to $Q$, and subsets of the domain assigned to the one-place predicate symbols $P_i$. In other words, the components of such a structure are exactly those of a model for $\varphi$: the domain is the set of worlds, the two-place relation assigned to $Q$ is the accessibility relation, and the subsets assigned to $P_i$ are just the assignments $V(p_i)$. It won't surprise that satisfaction of $\varphi$ in a modal model and of $\mathrm{ST}_x(\varphi)$ in the corresponding structure agree:

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**Proposition frd.2.** *Let $\mathfrak{M} = \langle W, R, V\rangle$, $\mathfrak{M}'$ be the first-order structure with $|\mathfrak{M}'| = W$, $Q^{\mathfrak{M}} = R$, and $P_i^{\mathfrak{M}'} = V(p_i)$, and $s(x) = w$. Then*

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', s \vDash \mathrm{ST}_x(\varphi)$$

*Proof.* By induction on $\varphi$. □

**Proposition frd.3.** *Suppose $\varphi$ is a modal formula and $\mathfrak{F} = \langle W, R \rangle$ is a frame. Let $\mathfrak{F}'$ be the first-order structure with $|\mathfrak{F}'| = W$ and $Q^{\mathfrak{F}'} = R$, and let $\varphi'$ be the second-order formula*

$$\forall X_1 \dots \forall X_n \, \forall x \, \mathrm{ST}_x(\varphi)[X_1/P_1, \dots, X_n/P_n],$$

*where $P_1$, …, $P_n$ are all one-place predicate symbols in $\mathrm{ST}_x(\varphi)$. Then*

$$\mathfrak{F} \vDash \varphi \text{ iff } \mathfrak{F}' \vDash \varphi'$$

*Proof.* $\mathfrak{F}' \vDash \varphi'$ iff for every structure $\mathfrak{M}'$ where $P_i^{\mathfrak{M}'} \subseteq W$ for $i = 1, \dots, n$, and for every $s$ with $s(x) \in W$, $\mathfrak{M}', s \vDash \mathrm{ST}_x(\varphi)$. By Proposition frd.2, that is the case iff for all models $\mathfrak{M}$ based on $\mathfrak{F}$ and every world $w \in W$, $\mathfrak{M}, w \Vdash \varphi$, i.e., $\mathfrak{F} \vDash \varphi$. □

**Definition frd.4.** A class $\mathcal{C}$ of frames is *second-order definable* if there is a sentence $\varphi$ in the second-order language with a single two-place predicate symbol $P$ and quantifiers only over monadic set variables such that $\mathfrak{F} = \langle W, R \rangle \in \mathcal{C}$ iff $\mathfrak{M} \vDash \varphi$ in the structure $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $P^{\mathfrak{M}} = R$.

**Corollary frd.5.** *If a class of frames is definable by a formula $\varphi$, the corresponding class of accessibility relations is definable by a monadic second-order sentence.*

*Proof.* The monadic second-order sentence $\varphi'$ of the preceding proof has the required property. □

As an example, consider again the formula $\Box p \to p$. It defines reflexivity. Reflexivity is of course first-order definable by the sentence $Q(x, x)$. But it is also definable by the monadic second-order sentence

$$\forall X \, \forall x \, (\forall y \, (Q(x, y) \to X(y)) \to X(x)).$$

This means, of course, that the two sentences are equivalent. Here's how you might convince yourself of this directly: First suppose the second-order sentence is true in a structure $M$. Since $x$ and $X$ is universally quantified, the remainder must hold for any $x \in W$ and set $X \subseteq W$, e.g, the set $\{z : Rxz\}$ where $R = Q^{\mathfrak{M}}$. So, for any $s$ with $s(x) \in W$ and $s(X) = \{z : Rzx\}$ we have $\mathfrak{M} \vDash \forall y \, (Q(x, y) \to X(y)) \to X(x)$. But by the way we've picked $s(X)$ that means $\mathfrak{M}, s \vDash \forall y \, (Q(x, y) \to Q(x, y)) \to Q(x, x)$, which is equivalent to $Q(x, x)$ since the antecedent is valid. Since $s(x)$ is arbitrary, we have $\mathfrak{M} \vDash \forall x \, Q(x, x)$.

Now suppose that $\mathfrak{M} \vDash Q(x, x)$ and show that $\mathfrak{M} \vDash \forall X \, \forall x \, (\forall y \, (Q(x, y) \to X(y)) \to X(x))$. Pick any assignment $s$, and assume $\mathfrak{M}, s \vDash \forall y \, (Q(x, y) \to X(y))$. Let $s'$ be the $y$-variant of $s$ with $s'(y) = x$; we have $\mathfrak{M}, s' \vDash Q(x, y) \to X(y))$, i.e., $\mathfrak{M}, s \vDash Q(x, x) \to X(x))$. Since $\mathfrak{M} \vDash \forall x \, Q(x, x)$, the antecedent is true, and we have $\mathfrak{M}, s \vDash X(x)$, which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic-second order sentence of the form $\varphi'$ is equivalent to a first-order sentence. There is no effective method to decide which ones are.

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## Bibliography