second-order-definability rev: 074a3f1 (2018-11-13) by OLP / CC–BY

Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal formula is true at a world are related to first-order formulas. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place predicate symbol $Q$ for the accessibility relation, but also a one-place predicate symbol $P_i$ for the propositional variables $p_i$ occurring in $\varphi$.

**Definition frd.1.** The standard translation $ST_x(\varphi)$ is inductively defined as follows:

1. $\varphi \equiv \bot$: $ST_x(\varphi) = \bot$.
2. $\varphi \equiv \top$: $ST_x(\varphi) = \top$.
3. $\varphi \equiv p_i$: $ST_x(\varphi) = P_i(x)$.
4. $\varphi \equiv \neg \psi$: $ST_x(\varphi) = \neg ST_x(\psi)$.
5. $\varphi \equiv (\psi \land \chi)$: $ST_x(\varphi) = (ST_x(\psi) \land ST_x(\chi))$.
6. $\varphi \equiv (\psi \lor \chi)$: $ST_x(\varphi) = (ST_x(\psi) \lor ST_x(\chi))$.
7. $\varphi \equiv (\psi \rightarrow \chi)$: $ST_x(\varphi) = (ST_x(\psi) \rightarrow ST_x(\chi))$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $ST_x(\varphi) = (ST_x(\psi) \leftrightarrow ST_x(\chi))$.
9. $\varphi \equiv \Box \psi$: $ST_x(\varphi) = \forall y (Q(x,y) \rightarrow ST_y(\psi))$.
10. $\varphi \equiv \Diamond \psi$: $ST_x(\varphi) = \exists y (Q(x,y) \land ST_y(\psi))$.

For instance, $ST_x(\square p \rightarrow p)$ is $\forall y (Q(x,y) \rightarrow P(y)) \rightarrow P(x)$. Any structure for the language of $ST_x(\varphi)$ requires a domain, a two-place relation assigned to $Q$, and subsets of the domain assigned to the one-place predicate symbols $P_i$. In other words, the components of such a structure are exactly those of a model for $\varphi$: the domain is the set of worlds, the two-place relation assigned to $Q$ is the accessibility relation, and the subsets assigned to $P_i$ are just the assignments $V(p_i)$. It won’t surprise that satisfaction of $\varphi$ in a modal model and of $ST_x(\varphi)$ in the corresponding structure agree:

**Proposition frd.2.** Let $\mathcal{M} = \langle W, R, V \rangle$, $\mathcal{M}'$ be the first-order structure with $|\mathcal{M}'| = W$, $Q^{\mathcal{M}'} = R$, and $P_i^{\mathcal{M}'} = V(p_i)$, and $s(x) = w$. Then $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', s \models ST_x(\varphi)$.

Proof. By induction on $\varphi$. □
Proposition frd.3. Suppose \( \varphi \) is a modal formula and \( \mathfrak{F} = (W, R) \) is a frame. Let \( \mathfrak{F}' = (W, R') \) be the first-order structure with \( |\mathfrak{F}'| = W \) and \( Q^{\mathfrak{F}'} = R' \), and let \( \varphi' \) be the second-order formula

\[
\forall X_1 \ldots \forall X_n \forall x \ ST_x(\varphi)[X_1/P_1, \ldots, X_n/P_n],
\]

where \( P_1, \ldots, P_n \) are all one-place predicate symbols in \( ST_x(\varphi) \). Then

\[
\mathfrak{F} \models \varphi \iff \mathfrak{F}' \models \varphi'
\]

Proof. \( \mathfrak{F}' \models \varphi' \iff \) for every structure \( \mathfrak{M}' \) where \( P_i^{\mathfrak{M}'} \subseteq W \) for \( i = 1, \ldots, n \), and for every \( s \) with \( s(x) \in W \), \( \mathfrak{M}', s \models ST_x(\varphi) \). By Proposition frd.2, that is the case iff for all models \( \mathfrak{M}' \) based on \( \mathfrak{F} \) and every world \( w \in W \), \( \mathfrak{M}, w \models \varphi \), i.e., \( \mathfrak{F} \models \varphi \).

Definition frd.4. A class \( C \) of frames is second-order definable if there is a sentence \( \varphi \) in the second-order language with a single two-place predicate symbol \( P \) and quantifiers only over monadic set variables such that \( \mathfrak{F} = (W, R) \in C \) iff \( \mathfrak{M} \models \varphi \) in the structure \( \mathfrak{M} \) with \( |\mathfrak{M}| = W \) and \( P^{\mathfrak{M}} = R \).

Corollary frd.5. If a class of frames is definable by a formula \( \varphi \), the corresponding class of accessibility relations is definable by a monadic second-order sentence.

Proof. The monadic second-order sentence \( \varphi' \) of the preceding proof has the required property.

As an example, consider again the formula \( \Box p \rightarrow p \). It defines reflexivity. Reflexivity is of course first-order definable by the sentence \( Q(x, x) \). But it is also definable by the monadic second-order sentence

\[
\forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)).
\]

This means, of course, that the two sentences are equivalent. Here’s how you might convince yourself of this directly: First suppose the second-order sentence is true in a structure \( M \). Since \( x \) and \( X \) is universally quantified, the remainder must hold for any \( x \in W \) and set \( X \subseteq W \), e.g., the set \( \{z : Rxz\} \) where \( R = Q^{\mathfrak{M}} \). So, for any \( s \) with \( s(x) \in W \) and \( s(X) = \{z : Rxz\} \) we have \( \mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x) \). But by the way we’ve picked \( s(X) \) that means \( \mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow Q(x, y)) \rightarrow Q(x, x) \), which is equivalent to \( Q(x, x) \) since the antecedent is valid. Since \( s(x) \) is arbitrary, we have \( \mathfrak{M} \models \forall x Q(x, x) \).

Now suppose that \( \mathfrak{M} \models Q(x, x) \) and show that \( \mathfrak{M} \models \forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)) \). Pick any assignment \( s \), and assume \( \mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y)) \). Let \( s' \) be the \( y \)-variant of \( s \) with \( s'(y) = x \); we have \( \mathfrak{M}, s' \models Q(x, y) \rightarrow X(y) \), i.e., \( \mathfrak{M}, s \models Q(x, x) \rightarrow X(x) \). Since \( \mathfrak{M} \models \forall x Q(x, x) \), the antecedent is true, and we have \( \mathfrak{M}, s \models X(x) \), which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic-second order sentence of the form \( \varphi' \) is equivalent to a first-order sentence. There is no effective method to decide which ones are.