

frd.1 Second-order Definability

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sec Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal **formula** is true at a world are related to first-order **formulas**. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place **predicate symbol** Q for the accessibility relation, but also a one-place **predicate symbol** P_i for the **propositional variables** p_i occurring in φ .

Definition frd.1. The *standard translation* $ST_x(\varphi)$ is inductively defined as follows:

1. $\varphi \equiv \perp$: $ST_x(\varphi) = \perp$.
2. $\varphi \equiv \top$: $ST_x(\varphi) = \top$.
3. $\varphi \equiv p_i$: $ST_x(\varphi) = P_i(x)$.
4. $\varphi \equiv \neg\psi$: $ST_x(\varphi) = \neg ST_x(\psi)$.
5. $\varphi \equiv (\psi \wedge \chi)$: $ST_x(\varphi) = (ST_x(\psi) \wedge ST_x(\chi))$.
6. $\varphi \equiv (\psi \vee \chi)$: $ST_x(\varphi) = (ST_x(\psi) \vee ST_x(\chi))$.
7. $\varphi \equiv (\psi \rightarrow \chi)$: $ST_x(\varphi) = (ST_x(\psi) \rightarrow ST_x(\chi))$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $ST_x(\varphi) = (ST_x(\psi) \leftrightarrow ST_x(\chi))$.
9. $\varphi \equiv \Box\psi$: $ST_x(\varphi) = \forall y (Q(x, y) \rightarrow ST_y(\psi))$.
10. $\varphi \equiv \Diamond\psi$: $ST_x(\varphi) = \exists y (Q(x, y) \wedge ST_y(\psi))$.

For instance, $ST_x(\Box p \rightarrow p)$ is $\forall y (Q(x, y) \rightarrow P(y)) \rightarrow P(x)$. Any **structure** for the language of $ST_x(\varphi)$ requires a domain, a two-place relation assigned to Q , and subsets of the domain assigned to the one-place **predicate symbols** P_i . In other words, the components of such a **structure** are exactly those of a model for φ : the domain is the set of worlds, the two-place relation assigned to Q is the accessibility relation, and the subsets assigned to P_i are just the assignments $V(p_i)$. It won't surprise that satisfaction of φ in a modal model and of $ST_x(\varphi)$ in the corresponding **structure** agree:

mod:frd:st:
prop:st **Proposition frd.2.** Let $\mathfrak{M} = \langle W, R, V \rangle$, \mathfrak{M}' be the first-order **structure** with $|\mathfrak{M}'| = W$, $Q^{\mathfrak{M}'} = R$, and $P_i^{\mathfrak{M}'} = V(p_i)$, and $s(x) = w$. Then

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', s \models ST_x(\varphi)$$

Proof. By induction on φ . □

Proposition frd.3. Suppose φ is a modal *formula* and $\mathfrak{F} = \langle W, R \rangle$ is a frame. Let \mathfrak{F}' be the first-order *structure* with $|\mathfrak{F}'| = W$ and $Q^{\mathfrak{F}'} = R$, and let φ' be the second-order *formula*

$$\forall X_1 \dots \forall X_n \forall x \text{ST}_x(\varphi)[X_1/P_1, \dots, X_n/P_n],$$

where P_1, \dots, P_n are all one-place *predicate symbols* in $\text{ST}_x(\varphi)$. Then

$$\mathfrak{F} \models \varphi \text{ iff } \mathfrak{F}' \models \varphi'$$

Proof. $\mathfrak{F}' \models \varphi'$ iff for every *structure* \mathfrak{M}' where $P_i^{\mathfrak{M}'} \subseteq W$ for $i = 1, \dots, n$, and for every s with $s(x) \in W$, $\mathfrak{M}', s \models \text{ST}_x(\varphi)$. By [Proposition frd.2](#), that is the case iff for all models \mathfrak{M} based on \mathfrak{F} and every world $w \in W$, $\mathfrak{M}, w \Vdash \varphi$, i.e., $\mathfrak{F} \models \varphi$. \square

Definition frd.4. A class \mathcal{C} of frames is *second-order definable* if there is a *sentence* φ in the second-order language with a single two-place *predicate symbol* P and quantifiers only over monadic set variables such that $\mathfrak{F} = \langle W, R \rangle \in \mathcal{C}$ iff $\mathfrak{M} \models \varphi$ in the *structure* \mathfrak{M} with $|\mathfrak{M}| = W$ and $P^{\mathfrak{M}} = R$.

Corollary frd.5. If a class of frames is definable by a *formula* φ , the corresponding class of accessibility relations is definable by a monadic second-order *sentence*.

Proof. The monadic second-order sentence φ' of the preceding proof has the required property. \square

As an example, consider again the *formula* $\Box p \rightarrow p$. It defines reflexivity. Reflexivity is of course first-order definable by the *sentence* $Q(x, x)$. But it is also definable by the monadic second-order *sentence*

$$\forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)).$$

This means, of course, that the two sentences are equivalent. Here's how you might convince yourself of this directly: First suppose the second-order sentence is true in a *structure* M . Since x and X is universally quantified, the remainder must hold for any $x \in W$ and set $X \subseteq W$, e.g, the set $\{z : Rzx\}$ where $R = Q^{\mathfrak{M}}$. So, for any s with $s(x) \in W$ and $s(X) = \{z : Rzx\}$ we have $\mathfrak{M} \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)$. But by the way we've picked $s(X)$ that means $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow Q(x, y)) \rightarrow Q(x, x)$, which is equivalent to $Q(x, x)$ since the antecedent is valid. Since $s(x)$ is arbitrary, we have $\mathfrak{M} \models \forall x Q(x, x)$.

Now suppose that $\mathfrak{M} \models Q(x, x)$ and show that $\mathfrak{M} \models \forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x))$. Pick any assignment s , and assume $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y))$. Let s' be the y -variant of s with $s'(y) = x$; we have $\mathfrak{M}, s' \models Q(x, y) \rightarrow X(y)$, i.e., $\mathfrak{M}, s \models Q(x, x) \rightarrow X(x)$. Since $\mathfrak{M} \models \forall x Q(x, x)$, the antecedent is true, and we have $\mathfrak{M}, s \models X(x)$, which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic-second order *sentence* of the form φ' is equivalent to a first-order *sentence*. There is no effective method to decide which ones are.

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Bibliography