

## frd.1 Second-order Definability

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Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal formula is true at a world are related to first-order formulas. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place predicate symbol  $Q$  for the accessibility relation, but also a one-place predicate symbol  $P_i$  for the propositional variables  $p_i$  occurring in  $\varphi$ .

**Definition frd.1.** The *standard translation*  $ST_x(\varphi)$  is inductively defined as follows:

1.  $\varphi \equiv \perp$ :  $ST_x(\varphi) = \perp$ .
2.  $\varphi \equiv \top$ :  $ST_x(\varphi) = \top$ .
3.  $\varphi \equiv p_i$ :  $ST_x(\varphi) = P_i(x)$ .
4.  $\varphi \equiv \neg\psi$ :  $ST_x(\varphi) = \neg ST_x(\psi)$ .
5.  $\varphi \equiv (\psi \wedge \chi)$ :  $ST_x(\varphi) = (ST_x(\psi) \wedge ST_x(\chi))$ .
6.  $\varphi \equiv (\psi \vee \chi)$ :  $ST_x(\varphi) = (ST_x(\psi) \vee ST_x(\chi))$ .
7.  $\varphi \equiv (\psi \rightarrow \chi)$ :  $ST_x(\varphi) = (ST_x(\psi) \rightarrow ST_x(\chi))$ .
8.  $\varphi \equiv (\psi \leftrightarrow \chi)$ :  $ST_x(\varphi) = (ST_x(\psi) \leftrightarrow ST_x(\chi))$ .
9.  $\varphi \equiv \Box\psi$ :  $ST_x(\varphi) = \forall y (Q(x, y) \rightarrow ST_y(\psi))$ .
10.  $\varphi \equiv \Diamond\psi$ :  $ST_x(\varphi) = \exists y (Q(x, y) \wedge ST_y(\psi))$ .

For instance,  $ST_x(\Box p \rightarrow p)$  is  $\forall y (Q(x, y) \rightarrow P(y)) \rightarrow P(x)$ . Any structure for the language of  $ST_x(\varphi)$  requires a domain, a two-place relation assigned to  $Q$ , and subsets of the domain assigned to the one-place predicate symbols  $P_i$ . In other words, the components of such a structure are exactly those of a model for  $\varphi$ : the domain is the set of worlds, the two-place relation assigned to  $Q$  is the accessibility relation, and the subsets assigned to  $P_i$  are just the assignments  $V(p_i)$ . It won't surprise that satisfaction of  $\varphi$  in a modal model and of  $ST_x(\varphi)$  in the corresponding structure agree:

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**Proposition frd.2.** Let  $\mathfrak{M} = \langle W, R, V \rangle$ ,  $\mathfrak{M}'$  be the first-order structure with  $|\mathfrak{M}'| = W$ ,  $Q^{\mathfrak{M}} = R$ , and  $P_i^{\mathfrak{M}'} = V(p_i)$ , and  $s(x) = w$ . Then

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', s \models ST_x(\varphi)$$

*Proof.* By induction on  $\varphi$ . □

**Proposition frd.3.** Suppose  $\varphi$  is a modal *formula* and  $\mathfrak{F} = \langle W, R \rangle$  is a frame. Let  $\mathfrak{F}'$  be the first-order *structure* with  $|\mathfrak{F}'| = W$  and  $Q^{\mathfrak{F}'} = R$ , and let  $\varphi'$  be the second-order *formula*

$$\forall X_1 \dots \forall X_n \forall x \text{ST}_x(\varphi)[X_1/P_1, \dots, X_n/P_n],$$

where  $P_1, \dots, P_n$  are all one-place *predicate symbols* in  $\text{ST}_x(\varphi)$ . Then

$$\mathfrak{F} \models \varphi \text{ iff } \mathfrak{F}' \models \varphi'$$

*Proof.*  $\mathfrak{F}' \models \varphi'$  iff for every *structure*  $\mathfrak{M}'$  where  $P_i^{\mathfrak{M}'} \subseteq W$  for  $i = 1, \dots, n$ , and for every  $s$  with  $s(x) \in W$ ,  $\mathfrak{M}', s \models \text{ST}_x(\varphi)$ . By [Proposition frd.2](#), that is the case iff for all models  $\mathfrak{M}$  based on  $\mathfrak{F}$  and every world  $w \in W$ ,  $\mathfrak{M}, w \Vdash \varphi$ , i.e.,  $\mathfrak{F} \models \varphi$ .  $\square$

**Definition frd.4.** A class  $\mathcal{C}$  of frames is *second-order definable* if there is a *sentence*  $\varphi$  in the second-order language with a single two-place *predicate symbol*  $P$  and quantifiers only over monadic set variables such that  $\mathfrak{F} = \langle W, R \rangle \in \mathcal{C}$  iff  $\mathfrak{M} \models \varphi$  in the *structure*  $\mathfrak{M}$  with  $|\mathfrak{M}| = W$  and  $P^{\mathfrak{M}} = R$ .

**Corollary frd.5.** If a class of frames is definable by a *formula*  $\varphi$ , the corresponding class of accessibility relations is definable by a monadic second-order *sentence*.

*Proof.* The monadic second-order sentence  $\varphi'$  of the preceding proof has the required property.  $\square$

As an example, consider again the *formula*  $\Box p \rightarrow p$ . It defines reflexivity. Reflexivity is of course first-order definable by the *sentence*  $Q(x, x)$ . But it is also definable by the monadic second-order *sentence*

$$\forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)).$$

This means, of course, that the two sentences are equivalent. Here's how you might convince yourself of this directly: First suppose the second-order sentence is true in a *structure*  $M$ . Since  $x$  and  $X$  is universally quantified, the remainder must hold for any  $x \in W$  and set  $X \subseteq W$ , e.g, the set  $\{z : Rzx\}$  where  $R = Q^{\mathfrak{M}}$ . So, for any  $s$  with  $s(x) \in W$  and  $s(X) = \{z : Rzx\}$  we have  $\mathfrak{M} \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)$ . But by the way we've picked  $s(X)$  that means  $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow Q(x, y)) \rightarrow Q(x, x)$ , which is equivalent to  $Q(x, x)$  since the antecedent is valid. Since  $s(x)$  is arbitrary, we have  $\mathfrak{M} \models \forall x Q(x, x)$ .

Now suppose that  $\mathfrak{M} \models Q(x, x)$  and show that  $\mathfrak{M} \models \forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x))$ . Pick any assignment  $s$ , and assume  $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y))$ . Let  $s'$  be the  $y$ -variant of  $s$  with  $s'(y) = x$ ; we have  $\mathfrak{M}, s' \models Q(x, y) \rightarrow X(y)$ , i.e.,  $\mathfrak{M}, s \models Q(x, x) \rightarrow X(x)$ . Since  $\mathfrak{M} \models \forall x Q(x, x)$ , the antecedent is true, and we have  $\mathfrak{M}, s \models X(x)$ , which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic-second order *sentence* of the form  $\varphi'$  is equivalent to a first-order *sentence*. There is no effective method to decide which ones are.

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**Bibliography**