If R is ...	*then ... is true in $\mathfrak{M}$:*	
serial: $\forall u \exists v Ruv$	$\Box p \to \Diamond p$	(D)
reflexive: $\forall w Rww$	$\Box p \to p$	(T)
symmetric: $\forall u \forall v (Ruv \to Rvu)$	$p \to \Box\Diamond p$	(B)
transitive: $\forall u \forall v \forall w ((Ruv \land Rvw) \to Ruw)$	$\Box p \to \Box\Box p$	(4)
euclidean: $\forall w \forall u \forall v ((Rwu \land Rwv) \to Ruv$	$\Diamond p \to \Box\Diamond p$	(5)

Table 1: Five correspondence facts.

mod:frd:acc: tab:five

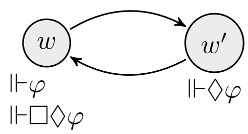

Figure 1: The argument from symmetry.

mod:frd:acc: fig:Bsymm

frd.1 Properties of Accessibility Relations

mod:frd:acc: sec

Many modal formulas turn out to be characteristic of simple, and even familiar, properties of the accessibility relation. In one direction, that means that any model that has a given property makes a corresponding formula (and all its substitution instances) true. We begin with five classical examples of kinds of accessibility relations and the formulas the truth of which they guarantee.

mod:frd:acc: thm:soundschemas

Theorem frd.1. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model. If R has the property on the left side of table 1, every instance of the formula on the right side is true in $\mathfrak{M}$.*

Proof. Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true all worlds in the model. So let $\varphi \to \Box\Diamond\varphi$ be a given instance of B, and let $w \in W$ be an arbitrary world. Suppose the antecedent φ is true at w, in order to show that $\Box\Diamond\varphi$ is true at w. So we need to show that $\Diamond\varphi$ is true at all w' accessible from w. Now, for any w' such that Rww' we have, using the hypothesis of symmetry, that also $Rw'w$ (see Figure 1). Since $\mathfrak{M}, w \Vdash \varphi$, we have $\mathfrak{M}, w' \Vdash \Diamond\varphi$. Since w' was an arbitrary world such that Rww', we have $\mathfrak{M}, w \Vdash \Box\Diamond\varphi$.

We leave the other cases as exercises. □

Problem frd.1. Complete the proof of Theorem frd.1

Notice that the converse implications of Theorem frd.1 do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property. In the case of T and reflexive models, it is easy to give an example of a model in which T itself fails: let $W = \{w\}$ and $V(p) = \emptyset$. Then R is not reflexive, but $\mathfrak{M}, w \Vdash \Box p$ and $\mathfrak{M}, w \nVdash p$. But here we have just a single instance of T that fails in $\mathfrak{M}$, other instances, e.g., $\Box\neg p \to \neg p$ are true. It is harder to give examples where *every substitution instance* of T is true in $\mathfrak{M}$ and $\mathfrak{M}$ is not reflexive. But there are such models, too:

Proposition frd.2. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model such that $W = \{u, v\}$, where worlds u and v are related by R: i.e., both Ruv and Rvu. Suppose that for all p: $u \in V(p) \Leftrightarrow v \in V(p)$. Then:* mod:frd:acc: prop:reflexive

1. *For all φ: $\mathfrak{M}, u \Vdash \varphi$ if and only if $\mathfrak{M}, v \Vdash \varphi$ (use induction on φ).*
2. *Every instance of* T *is true in $\mathfrak{M}$.*

Since $\mathfrak{M}$ is not reflexive (it is, in fact, irreflexive*), the converse of Theorem frd.1 fails in the case of* T *(similar arguments can be given for some—though not all—the other schemas mentioned in Theorem frd.1).*

Problem frd.2. Prove the claims in Proposition frd.2.

Although we will focus on the five classical formulas D, T, B, 4, and 5, we record in table 2 a few more properties of accessibility relations. The accessibility relation R is partially functional, if from every world at most one world is accessible. If it is the case that from every world exactly one world is accessible, we call it functional. (Thus the functional relations are precisely those that are both serial and partially functional). They are called "functional" because the accessibility relation operates like a (partial) function. A relation is weakly dense if whenever Ruv, there is a w "between" u and v. So weakly dense relations are in a sense the opposite of transitive relations: in a transitive relation, whenever you can reach v from u by a detour via w, you can reach v from u directly; in a weakly dense relation, whenever you can reach v from u directly, you can also reach it by a detour via some w. A relation is weakly directed if whenever you can reach worlds u and v from some world w, you can reach a single world t from both u and v—this is sometimes called the "diamond property" or "confluence."

Problem frd.3. Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model. Show that if R satisfies the left-hand properties of table 2, every instance of the corresponding right-hand formula is true in $\mathfrak{M}$.

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Bibliography

If R is ...	*then ... is true in $\mathfrak{M}$:*
partially functional: $\forall w \forall u \forall v((Rwu \land Rwv) \rightarrow u = v)$	$\Diamond p \rightarrow \Box p$
functional: $\forall w \exists v \forall u(Rwu \leftrightarrow u = v)$	$\Diamond p \leftrightarrow \Box p$
weakly dense: $\forall u \forall v(Ruv \rightarrow \exists w(Ruw \land Rwv))$	$\Box\Box p \rightarrow \Box p$
weakly connected: $\forall w \forall u \forall v((Rwu \land Rwv) \rightarrow (Ruv \lor u = v \lor Rvu))$	$\Box((p \land \Box p) \rightarrow q) \lor \Box((q \land \Box q) \rightarrow p)$ (L)
weakly directed: $\forall w \forall u \forall v((Rwu \land Rwv) \rightarrow \exists t(Rut \land Rvt)$	$\Diamond\Box p \rightarrow \Box\Diamond p$ (G)

Table 2: Five more correspondence facts.

mod:frd:acc:
tab:anotherfive