frd.1 Introduction

nml:frd:int:sec One question that interests modal logicians is the relationship between the accessibility relation and the truth of certain formulas in models with that accessibility relation. For instance, suppose the accessibility relation is reflexive, i.e., for every $w \in W$, Rww. In other words, every world is accessible from itself. That means that when $\Box\varphi$ is true at a world w, w itself is among the accessible worlds at which φ must therefore be true. So, if the accessibility relation R of $\mathfrak{M}$ is reflexive, then whatever world w and formula φ we take, $\Box\varphi \to \varphi$ will be true there (in other words, the schema $\Box p \to p$ and all its substitution instances are true in $\mathfrak{M}$).

The converse, however, is false. It's not the case, e.g., that if $\Box p \to p$ is true in $\mathfrak{M}$, then R is reflexive. For we can easily find a non-reflexive model $\mathfrak{M}$ where $\Box p \to p$ is true at all worlds: take the model with a single world w, not accessible from itself, but with $w \in V(p)$. By picking the truth value of p suitably, we can make $\Box\varphi \to \varphi$ true in a model that is not reflexive.

The solution is to remove the variable assignment V from the equation. If we require that $\Box p \to p$ is true at all worlds in $\mathfrak{M}$, regardless of which worlds are in $V(p)$, then it is necessary that R is reflexive. For in any non-reflexive model, there will be at least one world w such that not Rww. If we set $V(p) = W \setminus \{w\}$, then p will be true at all worlds other than w, and so at all worlds accessible from w (since w is guaranteed not to be accessible from w, and w is the only world where p is false). On the other hand, p is false at w, so $\Box p \to p$ is false at w.

This suggests that we should introduce a notation for model structures without a valuation: we call these *frames*. A frame $\mathfrak{F}$ is simply a pair $\langle W, R\rangle$ consisting of a set of worlds with an accessibility relation. Every model $\langle W, R, V\rangle$ is then, as we say, *based on* the frame $\langle W, R\rangle$. Conversely, a frame determines the class of models based on it; and a class of frames determines the class of models which are based on any frame in the class. And we can define $\mathfrak{F} \vDash \varphi$, the notion of a formula being *valid* in a frame as: $\mathfrak{M} \Vdash \varphi$ for all $\mathfrak{M}$ based on $\mathfrak{F}$.

With this notation, we can establish correspondence relations between formulas and classes of frames: e.g., $\mathfrak{F} \vDash \Box p \to p$ if, and only if, $\mathfrak{F}$ is reflexive.

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Bibliography