

## Chapter udf

# Frame Definability

### frd.1 Introduction

mod:frd:int:  
sec

One question that interests modal logicians is the relationship between the accessibility relation and the truth of certain **formulas** in models with that accessibility relation. For instance, suppose the accessibility relation is reflexive, i.e., for every  $w \in W$ ,  $Rww$ . In other words, every world is accessible from itself. That means that when  $\Box\varphi$  is true at a world  $w$ ,  $w$  itself is among the accessible worlds at which  $\varphi$  must therefore be true. So, if the accessibility relation  $R$  of  $\mathfrak{M}$  is reflexive, then whatever world  $w$  and formula  $\varphi$  we take,  $\Box\varphi \rightarrow \varphi$  will be true there (in other words, the schema  $\Box p \rightarrow p$  and all its substitution instances are true in  $\mathfrak{M}$ ).

The converse, however, is false. It's not the case, e.g., that if  $\Box p \rightarrow p$  is true in  $\mathfrak{M}$ , then  $R$  is reflexive. For we can easily find a non-reflexive model  $\mathfrak{M}$  where  $\Box p \rightarrow p$  is true at all worlds: take the model with a single world  $w$ , not accessible from itself, but with  $w \in V(p)$ . By picking the truth value of  $p$  suitably, we can make  $\Box\varphi \rightarrow \varphi$  true in a model that is not reflexive.

The solution is to remove the variable assignment  $V$  from the equation. If we require that  $\Box p \rightarrow p$  is true at all worlds in  $\mathfrak{M}$ , regardless of which worlds are in  $V(p)$ , then it is necessary that  $R$  is reflexive. For in any non-reflexive model, there will be at least one world  $w$  such that not  $Rww$ . If we set  $V(p) = W \setminus \{w\}$ , then  $p$  will be true at all worlds other than  $w$ , and so at all worlds accessible from  $w$  (since  $w$  is guaranteed not to be accessible from  $w$ , and  $w$  is the only world where  $p$  is false). On the other hand,  $p$  is false at  $w$ , so  $\Box p \rightarrow p$  is false at  $w$ .

This suggests that we should introduce a notation for model structures without a valuation: we call these *frames*. A frame  $\mathfrak{F}$  is simply a pair  $\langle W, R \rangle$  consisting of a set of worlds with an accessibility relation. Every model  $\langle W, R, V \rangle$  is then, as we say, *based on* the frame  $\langle W, R \rangle$ . Conversely, a frame determines the class of models based on it; and a class of frames determines the class of models which are based on any frame in the class. And we can define  $\mathfrak{F} \models \varphi$ , the notion of a **formula** being *valid* in a frame as:  $\mathfrak{M} \models \varphi$  for all  $\mathfrak{M}$  based on  $\mathfrak{F}$ .

If $R$ is ...	then ... is true in $\mathfrak{M}$ :
<i>serial</i> : $\forall u \exists v Ruv$	$\Box p \rightarrow \Diamond p$ (D)
<i>reflexive</i> : $\forall w Rww$	$\Box p \rightarrow p$ (T)
<i>symmetric</i> : $\forall u \forall v (Ruv \rightarrow Rvu)$	$p \rightarrow \Box \Diamond p$ (B)
<i>transitive</i> : $\forall u \forall v \forall w ((Ruv \wedge Rvw) \rightarrow Ruw)$	$\Box p \rightarrow \Box \Box p$ (4)
<i>euclidean</i> : $\forall w \forall u \forall v ((Rwu \wedge Rvw) \rightarrow Ruw)$	$\Diamond p \rightarrow \Box \Diamond p$ (5)

Table frd.1: Five correspondence facts.

mod:frd:acc:  
tab:five

With this notation, we can establish correspondence relations between **formulas** and classes of frames: e.g.,  $\mathfrak{F} \models \Box p \rightarrow p$  if, and only if,  $\mathfrak{F}$  is reflexive.

## frd.2 Properties of Accessibility Relations

Many modal **formulas** turn out to be characteristic of simple, and even familiar, properties of the accessibility relation. In one direction, that means that any model that has a given property makes a corresponding **formula** (and all its substitution instances) true. We begin with five classical examples of kinds of accessibility relations and the formulas the truth of which they guarantee.

mod:frd:acc:  
sec

**Theorem frd.1.** *Let  $\mathfrak{M} = \langle W, R, V \rangle$  be a model. If  $R$  has the property on the left side of table frd.1, every instance of the **formula** on the right side is true in  $\mathfrak{M}$ .*

mod:frd:acc:  
thm:soundschemas

*Proof.* Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true all worlds in the model. So let  $\varphi \rightarrow \Box \Diamond \varphi$  be a given instance of B, and let  $w \in W$  be an arbitrary world. Suppose the antecedent  $\varphi$  is true at  $w$ , in order to show that  $\Box \Diamond \varphi$  is true at  $w$ . So we need to show that  $\Diamond \varphi$  is true at all  $w'$  accessible from  $w$ . Now, for any  $w'$  such that  $Rww'$  we have, using the hypothesis of symmetry, that also  $Rw'w$  (see Figure frd.1). Since  $\mathfrak{M}, w \Vdash \varphi$ , we have  $\mathfrak{M}, w' \Vdash \Diamond \varphi$ . Since  $w'$  was an arbitrary world such that  $Rww'$ , we have  $\mathfrak{M}, w \Vdash \Box \Diamond \varphi$ .

We leave the other cases as exercises. □

**Problem frd.1.** Complete the proof of Theorem frd.1

Notice that the converse implications of Theorem frd.1 do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property. In the case of T and reflexive models, it is easy to give an example of a model in which T itself fails: let  $W = \{w\}$  and  $V(p) = \emptyset$ . Then  $R$  is not reflexive, but  $\mathfrak{M}, w \Vdash \Box p$  and  $\mathfrak{M}, w \not\Vdash p$ . But here we have just a single instance of T that fails in  $\mathfrak{M}$ , other instances, e.g.,  $\Box \neg p \rightarrow \neg p$

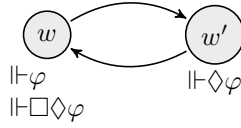


Figure frd.1: The argument from symmetry.

mod:frd:acc:  
fig:Bsymm

are true. It is harder to give examples where *every substitution instance* of  $T$  is true in  $\mathfrak{M}$  and  $\mathfrak{M}$  is not reflexive. But there are such models, too:

mod:frd:acc:  
prop:reflexive

**Proposition frd.2.** Let  $\mathfrak{M} = \langle W, R, V \rangle$  be a model such that  $W = \{u, v\}$ , where worlds  $u$  and  $v$  are related by  $R$ : i.e., both  $Ruv$  and  $Rvu$ . Suppose that for all  $p$ :  $u \in V(p) \Leftrightarrow v \in V(p)$ . Then:

1. For all  $\varphi$ :  $\mathfrak{M}, u \Vdash \varphi$  if and only if  $\mathfrak{M}, v \Vdash \varphi$  (use induction on  $\varphi$ ).
2. Every instance of  $T$  is true in  $\mathfrak{M}$ .

Since  $\mathfrak{M}$  is not reflexive (it is, in fact, irreflexive), the converse of *Theorem frd.1* fails in the case of  $T$  (similar arguments can be given for some—though not all—the other schemas mentioned in *Theorem frd.1*).

**Problem frd.2.** Prove the claims in *Proposition frd.2*.

Although we will focus on the five classical formulas  $D$ ,  $T$ ,  $B$ ,  $4$ , and  $5$ , we record in *table frd.2* a few more properties of accessibility relations. The accessibility relation  $R$  is partially functional, if from every world at most one world is accessible. If it is the case that from every world exactly one world is accessible, we call it functional. (Thus the functional relations are precisely those that are both serial and partially functional). They are called “functional” because the accessibility relation operates like a (partial) function. A relation is weakly dense if whenever  $Ruv$ , there is a  $w$  “between”  $u$  and  $v$ . So weakly dense relations are in a sense the opposite of transitive relations: in a transitive relation, whenever you can reach  $v$  from  $u$  by a detour via  $w$ , you can reach  $v$  from  $u$  directly; in a weakly dense relation, whenever you can reach  $v$  from  $u$  directly, you can also reach it by a detour via some  $w$ . A relation is weakly directed if whenever you can reach worlds  $u$  and  $v$  from some world  $w$ , you can reach a single world  $t$  from both  $u$  and  $v$ —this is sometimes called the “diamond property” or “confluence.”

**Problem frd.3.** Let  $\mathfrak{M} = \langle W, R, V \rangle$  be a model. Show that if  $R$  satisfies the left-hand properties of *table frd.2*, every instance of the corresponding right-hand formula is true in  $\mathfrak{M}$ .

If $R$ is ...	then ... is true in $\mathfrak{M}$ :
<i>partially functional</i> : $\forall w \forall u \forall v ((Rwu \wedge Rvw) \rightarrow u = v)$	$\Diamond p \rightarrow \Box p$
<i>functional</i> : $\forall w \exists v \forall u (Rwu \leftrightarrow u = v)$	$\Diamond p \leftrightarrow \Box p$
<i>weakly dense</i> : $\forall u \forall v (Ruv \rightarrow \exists w (Ruw \wedge Rvw))$	$\Box \Box p \rightarrow \Box p$
<i>weakly connected</i> : $\forall w \forall u \forall v ((Rwu \wedge Rvw) \rightarrow (Rwu \vee u = v \vee Rvu))$	$\Box((p \wedge \Box p) \rightarrow q) \vee \Box((q \wedge \Box q) \rightarrow p)$ (L)
<i>weakly directed</i> : $\forall w \forall u \forall v ((Rwu \wedge Rvw) \rightarrow \exists t (Rut \wedge Rvt))$	$\Diamond \Box p \rightarrow \Box \Diamond p$ (G)

Table frd.2: Five more correspondence facts.

### frd.3 Frames

mod:frd:acc:  
tab:anotherfive

**Definition frd.3.** A *frame* is a pair  $\mathfrak{F} = \langle W, R \rangle$  where  $W$  is a non-empty set of worlds and  $R$  a binary relation on  $W$ . A model  $\mathfrak{M}$  is *based on* a frame  $\mathfrak{F} = \langle W, R \rangle$  if and only if  $\mathfrak{M} = \langle W, R, V \rangle$ .

mod:frd:fra:  
sec

**Definition frd.4.** If  $\mathfrak{F}$  is a frame, we say that  $\varphi$  is *valid in*  $\mathfrak{F}$ ,  $\mathfrak{F} \models \varphi$ , if  $\mathfrak{M} \Vdash \varphi$  for every model  $\mathfrak{M}$  based on  $\mathfrak{F}$ .

If  $\mathcal{F}$  is a class of frames, we say  $\varphi$  is *valid in*  $\mathcal{F}$ ,  $\mathcal{F} \models \varphi$ , iff  $\mathfrak{F} \models \varphi$  for every frame  $\mathfrak{F} \in \mathcal{F}$ .

The reason frames are interesting is that correspondence between schemas and properties of the accessibility relation  $R$  is at the level of frames, *not of models*. For instance, although T is true in all reflexive models, not every model in which T is true is reflexive. However, it *is* true that not only is T *valid* on all reflexive *frames*, also every frame in which T is valid is reflexive.

*Remark 1.* Validity in a class of frames is a special case of the notion of validity in a class of models:  $\mathcal{F} \models \varphi$  iff  $\mathcal{C} \models \varphi$  where  $\mathcal{C}$  is the class of all models based on a frame in  $\mathcal{F}$ .

Obviously, if a **formula** or a schema is valid, i.e., valid with respect to the class of *all* models, it is also valid with respect to any class  $\mathcal{F}$  of frames.

### frd.4 Frame Definability

Even though the converse implications of **Theorem frd.1** fail, they hold if we replace “model” by “frame”: for the properties considered in **Theorem frd.1**, it *is* true that if a **formula** is valid in a *frame* then the accessibility relation of that frame has the corresponding property. So, the **formulas** considered *define* the classes of frames that have the corresponding property.

mod:frd:def:  
sec

**Definition frd.5.** If  $\mathcal{C}$  is a class of frames, we say  $\varphi$  *defines*  $\mathcal{C}$  iff  $\mathfrak{F} \models \varphi$  for all and only frames  $\mathfrak{F} \in \mathcal{C}$ .

We now proceed to establish the full definability results for frames.

*mod:frd:def:*  
*thm:fullCorrespondence*

**Theorem frd.6.** *If the formula on the right side of table frd.1 is valid in a frame  $\mathfrak{F}$ , then  $\mathfrak{F}$  has the property on the left side.*

*Proof.* 1. Suppose D is valid in  $\mathfrak{F} = \langle W, R \rangle$ , i.e.,  $\mathfrak{F} \models \Box p \rightarrow \Diamond p$ . Let  $\mathfrak{M} = \langle W, R, V \rangle$  be a model based on  $\mathfrak{F}$ , and  $w \in W$ . We have to show that there is a  $v$  such that  $Rwv$ . Suppose not: then both  $\mathfrak{M} \Vdash \Box \varphi$  and  $\mathfrak{M}, w \not\models \Diamond \varphi$  for any  $\varphi$ , including  $p$ . But then  $\mathfrak{M}, w \not\models \Box p \rightarrow \Diamond p$ , contradicting the assumption that  $\mathfrak{F} \models \Box p \rightarrow \Diamond p$ .

2. Suppose T is valid in  $\mathfrak{F}$ , i.e.,  $\mathfrak{F} \models \Box p \rightarrow p$ . Let  $w \in W$  be an arbitrary world; we need to show  $Rww$ . Let  $u \in V(p)$  if and only if  $Rwu$  (when  $q$  is other than  $p$ ,  $V(q)$  is arbitrary, say  $V(q) = \emptyset$ ). Let  $\mathfrak{M} = \langle W, R, V \rangle$ . By construction, for all  $u$  such that  $Rwu$ :  $\mathfrak{M}, u \Vdash p$ , and hence  $\mathfrak{M}, w \Vdash \Box p$ . But by hypothesis  $\Box p \rightarrow p$  is true at  $w$ , so that  $\mathfrak{M}, w \Vdash p$ , but by definition of  $V$  this is possible only if  $Rww$ .

3. We prove the contrapositive: Suppose  $\mathfrak{F}$  is not symmetric, we show that B, i.e.,  $p \rightarrow \Box \Diamond p$  is not valid in  $\mathfrak{F} = \langle W, R \rangle$ . If  $\mathfrak{F}$  is not symmetric, there are  $u, v \in W$  such that  $Ruv$  but not  $Rvu$ . Define  $V$  such that  $w \in V(p)$  if and only if not  $Rvw$  (and  $V$  is arbitrary otherwise). Let  $\mathfrak{M} = \langle W, R, V \rangle$ . Now, by definition of  $V$ ,  $\mathfrak{M}, w \Vdash p$  for all  $w$  such that not  $Rvw$ , in particular,  $\mathfrak{M}, u \Vdash p$  since not  $Rvu$ . Also, since  $Rvw$  iff  $p \notin V(w)$ , there is no  $w$  such that  $Rvw$  and  $\mathfrak{M}, w \Vdash p$ , and hence  $\mathfrak{M}, v \not\models \Diamond p$ . Since  $Ruv$ , also  $\mathfrak{M}, u \not\models \Box \Diamond p$ . It follows that  $\mathfrak{M}, u \not\models p \rightarrow \Box \Diamond p$ , and so B is not valid in  $\mathfrak{F}$ .

4. Suppose 4 is valid in  $\mathfrak{F} = \langle W, R \rangle$ , i.e.,  $\mathfrak{F} \models \Box p \rightarrow \Box \Box p$ , and let  $u, v, w \in W$  be arbitrary worlds such that  $Ruv$  and  $Rvw$ ; we need to show that  $Ruw$ . Define  $V$  such that  $z \in V(p)$  if and only if  $Ruz$  (and  $V$  is arbitrary otherwise). Let  $\mathfrak{M} = \langle W, R, V \rangle$ . By definition of  $V$ ,  $\mathfrak{M}, z \Vdash p$  for all  $z$  such that  $Ruz$ , and hence  $\mathfrak{M}, u \Vdash \Box p$ . But by hypothesis 4,  $\Box p \rightarrow \Box \Box p$ , is true at  $u$ , so that  $\mathfrak{M}, u \Vdash \Box \Box p$ . Since  $Ruv$  and  $Rvw$ , we have  $\mathfrak{M}, w \Vdash p$ , but by definition of  $V$  this is possible only if  $Ruw$ , as desired.

5. We proceed contrapositively, assuming that the frame  $\mathfrak{F} = \langle W, R \rangle$  is not euclidean, and show that it falsifies 5, i.e., i.e.,  $\mathfrak{F} \not\models \Diamond p \rightarrow \Box \Diamond p$ . Suppose there are worlds  $u, v, w \in W$  such that  $Rwu$  and  $Rvw$  but not  $Ruv$ . Define  $V$  such that for all worlds  $z$ ,  $z \in V(p)$  if and only if it is *not* the case that  $Ruz$ . Let  $\mathfrak{M} = \langle W, R, V \rangle$ . Then by hypothesis  $\mathfrak{M}, v \Vdash p$  and since  $Rvw$  also  $\mathfrak{M}, w \Vdash \Diamond p$ . However, there is no world  $y$  such that  $Ruy$  and  $\mathfrak{M}, y \Vdash p$  so  $\mathfrak{M}, u \not\models \Box \Diamond p$ . Since  $Rwu$ , it follows that  $\mathfrak{M}, w \not\models \Diamond p \rightarrow \Box \Diamond p$ , so that 5,  $\Diamond p \rightarrow \Box \Diamond p$ , fails at  $w$ .

□

You'll notice a difference between the proof for D and the other cases: no mention was made of the valuation  $V$ . In effect, we proved that if  $\mathfrak{M} \Vdash D$  then  $\mathfrak{M}$  is serial. So D defines the class of serial *models*, not just frames.

**Corollary frd.7.** *Any model where D is true is serial.*

*mod:frd:def:  
prop:D-serial*

**Corollary frd.8.** *Each formula on the right side of table frd.1 defines the class of frames which have the property on the left side.*

*Proof.* In Theorem frd.1, we proved that if a model has the property on the left, the formula on the right is true in it. Thus, if a frame  $\mathfrak{F}$  has the property on the left, the formula on the left is valid in  $\mathfrak{F}$ . In Theorem frd.6, we proved the converse implications: if a formula on the right is valid in  $\mathfrak{F}$ ,  $\mathfrak{F}$  has the property on the left.  $\square$

**Problem frd.4.** Show that if the formula on the right side of table frd.2 is valid in a frame  $\mathfrak{F}$ , then  $\mathfrak{F}$  has the property on the left side. To do this, consider a frame that does *not* satisfy the property on the left, and define a suitable  $V$  such that the formula on the right is false at some world.

Theorem frd.6 also shows that the properties can be combined: for instance if both B and 4 are valid in  $\mathfrak{F}$  then the frame is both symmetric and transitive, etc. Many important modal logics are characterized as the set of formulas valid in all frames that combine some frame properties, and so we can characterize them as the set of formulas valid in all frames in which the corresponding defining formulas are valid. For instance, the classical system **S4** is the set of all formulas valid in all reflexive and transitive frames, i.e., in all those where both T and 4 are valid. **S5** is the set of all formulas valid in all reflexive, symmetric, and euclidean frames, i.e., all those where all of T, B, and 5 are valid.

Logical relationships between properties of  $R$  in general correspond to relationships between the corresponding defining formulas. For instance, every reflexive relation is serial; hence, whenever T is valid in a frame, so is D. (Note that this relationship is *not* that of entailment. It is not the case that whenever  $\mathfrak{M}, w \Vdash T$  then  $\mathfrak{M}, w \Vdash D$ .) We record some such relationships.

**Proposition frd.9.** *Let  $R$  be a binary relation on a set  $W$ ; then:*

*mod:frd:def:  
prop:relation-facts*

1. *If  $R$  is reflexive, then it is serial.*
2. *If  $R$  is symmetric, then it is transitive if and only if it is euclidean.*
3. *If  $R$  is symmetric or euclidean then it is weakly directed (it has the “diamond property”).*
4. *If  $R$  is euclidean then it is weakly connected.*
5. *If  $R$  is functional then it is serial.*

**Problem frd.5.** Prove Proposition frd.9.

## frd.5 First-order Definability

mod:frd:fol:  
sec

We've seen that a number of properties of accessibility relations of frames can be defined by modal **formulas**. For instance, symmetry of frames can be defined by the **formula**  $B$ ,  $p \rightarrow \Box \Diamond p$ . The conditions we've encountered so far can all be expressed by first-order **formulas** in a **language** involving a single two-place **predicate symbol**. For instance, symmetry is defined by  $\forall x \forall y (Q(x, y) \rightarrow Q(y, x))$  in the sense that a first-order **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}| = W$  and  $Q^{\mathfrak{M}} = R$  satisfies the preceding formula iff  $R$  is symmetric. This suggests the following definition:

**Definition frd.10.** A class  $\mathcal{C}$  of frames is *first-order definable* if there is a **sentence**  $\varphi$  in the first-order language with a single two-place **predicate symbol**  $P$  such that  $\mathfrak{F} = \langle W, R \rangle \in \mathcal{C}$  iff  $\mathfrak{M} \models \varphi$  in the first-order **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}| = W$  and  $Q^{\mathfrak{M}} = R$ .

It turns out that the properties and modal **formulas** that define them considered so far are exceptional. Not every **formula** defines a first-order definable class of frames, and not every first-order definable class of frames is definable by a modal formula.

A counterexample to the first is given by the Löb formula:

$$\Box(\Box p \rightarrow p) \rightarrow \Box p. \quad (\text{W})$$

W defines the class of transitive and converse well-founded frames. A relation is well-founded if there is no infinite sequence  $w_1, w_2, \dots$  such that  $Rw_2w_1, Rw_3w_2, \dots$ . For instance, the relation  $<$  on  $\mathbb{N}$  is well-founded, whereas the relation  $<$  on  $\mathbb{Z}$  is not. A relation is converse well-founded iff its converse is well-founded. So converse well-founded relations are those where there is no infinite sequence  $w_1, w_2, \dots$  such that  $Rw_1w_2, Rw_2w_3, \dots$ .

There is, however, no first-order **formula** defining transitive converse well-founded relations. For suppose  $\mathfrak{M} \models \beta$  iff  $R = Q^{\mathfrak{M}}$  is transitive converse well-founded. Let  $\varphi_n$  be the **formula**

$$(Q(a_1, a_2) \wedge \dots \wedge Q(a_{n-1}, a_n))$$

Now consider the set of **formulas**

$$\Gamma = \{\beta, \varphi_1, \varphi_2, \dots\}.$$

Every finite subset of  $\Gamma$  is satisfiable: Let  $k$  be largest such that  $\varphi_k$  is in the subset,  $|\mathfrak{M}_k| = \{1, \dots, k\}$ ,  $a_i^{\mathfrak{M}_k} = i$ , and  $P^{\mathfrak{M}_k} = <$ . Since  $<$  on  $\{1, \dots, k\}$  is transitive and converse well-founded,  $\mathfrak{M}_k \models \beta$ .  $\mathfrak{M}_k \models \varphi_i$  by construction, for all  $i \leq k$ . By the Compactness Theorem for first-order logic,  $\Gamma$  is satisfiable in some **structure**  $\mathfrak{M}$ . By hypothesis, since  $\mathfrak{M} \models \beta$ , the relation  $Q^{\mathfrak{M}}$  is converse well-founded. But clearly,  $a_1^{\mathfrak{M}}, a_2^{\mathfrak{M}}, \dots$  would form an infinite sequence of the kind ruled out by converse well-foundedness.

A counterexample to the second claim is given by the property of universality: for every  $u$  and  $v$ ,  $Ruv$ . Universal frames are first-order definable by

the formula  $\forall x \forall y Q(x, y)$ . However, no modal formula is valid in all and only the universal frames. This is a consequence of a result that is independently interesting: the formulas valid in universal frames are exactly the same as those valid in reflexive, symmetric, and transitive frames. There are reflexive, symmetric, and transitive frames that are not universal, hence every formula valid in all universal frames is also valid in some non-universal frames.

## frd.6 Equivalence Relations and S5

The modal logic **S5** is characterized as the set of formulas valid on all universal frames, i.e., every world is accessible from every world, including itself. In such a scenario,  $\Box$  corresponds to necessity and  $\Diamond$  to possibility:  $\Box\varphi$  is true if  $\varphi$  is true at every world, and  $\Diamond\varphi$  is true if  $\varphi$  is true at some world. It turns out that **S5** can also be characterized as the formulas valid on all reflexive, symmetric, and transitive frames, i.e., on all *equivalence relations*.

mod:frd:es5:  
sec

**Definition frd.11.** A binary relation  $R$  on  $W$  is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation  $R$  on  $W$  is *universal* if and only if  $Ruv$  for all  $u, v \in W$ .

Since T, B, and 4 characterize the reflexive, symmetric, and transitive frames, the frames where the accessibility relation is an equivalence relation are exactly those in which all four formulas are valid. It turns out that the equivalence relations can also be characterized by other combinations of formulas, since the conditions with which we've defined equivalence relations are equivalent to combinations of other familiar conditions on  $R$ .

**Proposition frd.12.** *The following are equivalent:*

mod:frd:es5:  
prop:equivalences

1.  $R$  is an equivalence relation;
2.  $R$  is reflexive and euclidean;
3.  $R$  is serial, symmetric, and euclidean;
4.  $R$  is serial, symmetric, and transitive.

*Proof.* Exercise. □

**Problem frd.6.** Prove Proposition frd.12 by showing:

1. If  $R$  is symmetric and transitive, it is euclidean.
2. If  $R$  is reflexive, it is serial.
3. If  $R$  is reflexive and euclidean, it is symmetric.
4. If  $R$  is symmetric and euclidean, it is transitive.
5. If  $R$  is serial, symmetric, and transitive, it is reflexive.



Explain why this suffices for the proof that the conditions are equivalent.

**Proposition frd.12** is the semantic counterpart to ??, in that it gives equivalent characterization of the modal logic of frames over which  $R$  is an equivalence (the logic traditionally referred to as **S5**).

What is the relationship between universal and equivalence relations? Although every universal relation is an equivalence relation, clearly not every equivalence relation is universal. However, the formulas valid on all universal relations are exactly the same as those valid on all equivalence relations.

**Proposition frd.13.** *Let  $R$  be an equivalence relation, and for each  $w \in W$  define the equivalence class of  $w$  as the set  $[w] = \{w' \in W : Rww'\}$ . Then:*

1.  $w \in [w]$ ;
2.  $R$  is universal on each equivalence class  $[w]$ ;
3. The collection of equivalence classes partitions  $W$  into mutually exclusive and jointly exhaustive subsets.

mod:frd:es5:  
prop:S5=univ **Proposition frd.14.** *A formula  $\varphi$  is valid in all frames  $\mathfrak{F} = \langle W, R \rangle$  where  $R$  is an equivalence relation, if and only if it is valid in all frames  $\mathfrak{F} = \langle W, R \rangle$  where  $R$  is universal. Hence, the logic of universal frames is just **S5**.*

*Proof.* It's immediate to verify that a universal relation  $R$  on  $W$  is an equivalence. Hence, if  $\varphi$  is valid in all frames where  $R$  is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose  $\psi$  is a formula that fails at a world  $w$  in a model  $\mathfrak{M} = \langle W, R, V \rangle$  based on a frame  $\langle W, R \rangle$ , where  $R$  is an equivalence on  $W$ . So  $\mathfrak{M}, w \not\models \psi$ . Define a model  $\mathfrak{M}' = \langle W', R', V' \rangle$  as follows:

1.  $W' = [w]$ ;
2.  $R'$  is universal on  $W'$ ;
3.  $V'(p) = V(p) \cap W'$ .

(So the set  $W'$  of worlds in  $\mathfrak{M}'$  is represented by the shaded area in [Figure frd.2](#).) It is easy to see that  $R$  and  $R'$  agree on  $W'$ . Then one can show by induction on formulas that for all  $w' \in W'$ :  $\mathfrak{M}', w' \models \varphi$  if and only if  $\mathfrak{M}, w' \models \varphi$  for each  $\varphi$  (this makes sense since  $W' \subseteq W$ ). In particular,  $\mathfrak{M}', w \not\models \psi$ , and  $\psi$  fails in a model based on a universal frame.  $\square$

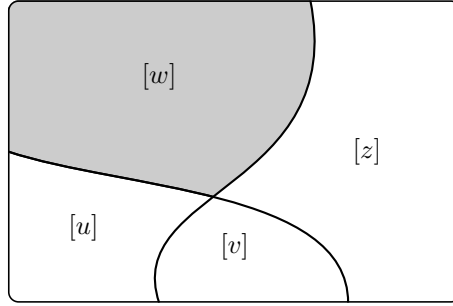


Figure frd.2: A partition of  $W$  in equivalence classes.

mod:frd:es5:  
fig:partition

## frd.7 Second-order Definability

Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal **formula** is true at a world are related to first-order **formulas**. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place **predicate symbol**  $Q$  for the accessibility relation, but also a one-place **predicate symbol**  $P_i$  for the **propositional variables**  $p_i$  occurring in  $\varphi$ .

mod:frd:st:  
sec

**Definition frd.15.** The *standard translation*  $ST_x(\varphi)$  is inductively defined as follows:

1.  $\varphi \equiv \perp$ :  $ST_x(\varphi) = \perp$ .
2.  $\varphi \equiv \top$ :  $ST_x(\varphi) = \top$ .
3.  $\varphi \equiv p_i$ :  $ST_x(\varphi) = P_i(x)$ .
4.  $\varphi \equiv \neg\psi$ :  $ST_x(\varphi) = \neg ST_x(\psi)$ .
5.  $\varphi \equiv (\psi \wedge \chi)$ :  $ST_x(\varphi) = (ST_x(\psi) \wedge ST_x(\chi))$ .
6.  $\varphi \equiv (\psi \vee \chi)$ :  $ST_x(\varphi) = (ST_x(\psi) \vee ST_x(\chi))$ .
7.  $\varphi \equiv (\psi \rightarrow \chi)$ :  $ST_x(\varphi) = (ST_x(\psi) \rightarrow ST_x(\chi))$ .
8.  $\varphi \equiv (\psi \leftrightarrow \chi)$ :  $ST_x(\varphi) = (ST_x(\psi) \leftrightarrow ST_x(\chi))$ .
9.  $\varphi \equiv \Box\psi$ :  $ST_x(\varphi) = \forall y (Q(x, y) \rightarrow ST_y(\psi))$ .
10.  $\varphi \equiv \Diamond\psi$ :  $ST_x(\varphi) = \exists y (Q(x, y) \wedge ST_y(\psi))$ .

For instance,  $\text{ST}_x(\Box p \rightarrow p)$  is  $\forall y (Q(x, y) \rightarrow P(y)) \rightarrow P(x)$ . Any **structure** for the language of  $\text{ST}_x(\varphi)$  requires a domain, a two-place relation assigned to  $Q$ , and subsets of the domain assigned to the one-place **predicate symbols**  $P_i$ . In other words, the components of such a **structure** are exactly those of a model for  $\varphi$ : the domain is the set of worlds, the two-place relation assigned to  $Q$  is the accessibility relation, and the subsets assigned to  $P_i$  are just the assignments  $V(\rho_i)$ . It won't surprise that satisfaction of  $\varphi$  in a modal model and of  $\text{ST}_x(\varphi)$  in the corresponding **structure** agree:

*mod:frd:st: prop:st* **Proposition frd.16.** *Let  $\mathfrak{M} = \langle W, R, V \rangle$ ,  $\mathfrak{M}'$  be the first-order **structure** with  $|\mathfrak{M}'| = W$ ,  $Q^{\mathfrak{M}'} = R$ , and  $P_i^{\mathfrak{M}'} = V(\rho_i)$ , and  $s(x) = w$ . Then*

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', s \models \text{ST}_x(\varphi)$$

*Proof.* By induction on  $\varphi$ . □

**Proposition frd.17.** *Suppose  $\varphi$  is a modal **formula** and  $\mathfrak{F} = \langle W, R \rangle$  is a **frame**. Let  $\mathfrak{F}'$  be the first-order **structure** with  $|\mathfrak{F}'| = W$  and  $Q^{\mathfrak{F}'} = R$ , and let  $\varphi'$  be the second-order **formula***

$$\forall X_1 \dots \forall X_n \forall x \text{ST}_x(\varphi)[X_1/P_1, \dots, X_n/P_n],$$

where  $P_1, \dots, P_n$  are all one-place **predicate symbols** in  $\text{ST}_x(\varphi)$ . Then

$$\mathfrak{F} \models \varphi \text{ iff } \mathfrak{F}' \models \varphi'$$

*Proof.*  $\mathfrak{F}' \models \varphi'$  iff for every **structure**  $\mathfrak{M}'$  where  $P_i^{\mathfrak{M}'} \subseteq W$  for  $i = 1, \dots, n$ , and for every  $s$  with  $s(x) \in W$ ,  $\mathfrak{M}', s \models \text{ST}_x(\varphi)$ . By **Proposition frd.16**, that is the case iff for all models  $\mathfrak{M}$  based on  $\mathfrak{F}$  and every world  $w \in W$ ,  $\mathfrak{M}, w \Vdash \varphi$ , i.e.,  $\mathfrak{F} \models \varphi$ . □

**Definition frd.18.** A class  $\mathcal{C}$  of frames is *second-order definable* if there is a **sentence**  $\varphi$  in the second-order language with a single two-place **predicate symbol**  $P$  and quantifiers only over monadic set variables such that  $\mathfrak{F} = \langle W, R \rangle \in \mathcal{C}$  iff  $\mathfrak{M} \models \varphi$  in the **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}| = W$  and  $P^{\mathfrak{M}} = R$ .

**Corollary frd.19.** *If a class of frames is definable by a **formula**  $\varphi$ , the corresponding class of accessibility relations is definable by a monadic second-order **sentence**.*

*Proof.* The monadic second-order sentence  $\varphi'$  of the preceding proof has the required property. □

As an example, consider again the **formula**  $\Box p \rightarrow p$ . It defines reflexivity. Reflexivity is of course first-order definable by the **sentence**  $Q(x, x)$ . But it is also definable by the monadic second-order **sentence**

$$\forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)).$$

This means, of course, that the two sentences are equivalent. Here's how you might convince yourself of this directly: First suppose the second-order sentence is true in a **structure**  $M$ . Since  $x$  and  $X$  is universally quantified, the remainder must hold for any  $x \in W$  and set  $X \subseteq W$ , e.g, the set  $\{z : Rzx\}$  where  $R = Q^{\mathfrak{M}}$ . So, for any  $s$  with  $s(x) \in W$  and  $s(X) = \{z : Rzx\}$  we have  $\mathfrak{M} \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)$ . But by the way we've picked  $s(X)$  that means  $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow Q(x, y)) \rightarrow Q(x, x)$ , which is equivalent to  $Q(x, x)$  since the antecedent is valid. Since  $s(x)$  is arbitrary, we have  $\mathfrak{M} \models \forall x Q(x, x)$ .

Now suppose that  $\mathfrak{M} \models Q(x, x)$  and show that  $\mathfrak{M} \models \forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x))$ . Pick any assignment  $s$ , and assume  $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y))$ . Let  $s'$  be the  $y$ -variant of  $s$  with  $s'(y) = x$ ; we have  $\mathfrak{M}, s' \models Q(x, y) \rightarrow X(y)$ , i.e.,  $\mathfrak{M}, s \models Q(x, x) \rightarrow X(x)$ . Since  $\mathfrak{M} \models \forall x Q(x, x)$ , the antecedent is true, and we have  $\mathfrak{M}, s \models X(x)$ , which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic-second order **sentence** of the form  $\varphi'$  is equivalent to a first-order **sentence**. There is no effective method to decide which ones are.

## Photo Credits

# Bibliography