

## frd.1 First-order Definability

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We've seen that a number of properties of accessibility relations of frames can be defined by modal **formulas**. For instance, symmetry of frames can be defined by the **formula**  $B, p \rightarrow \Box \Diamond p$ . The conditions we've encountered so far can all be expressed by first-order **formulas** in a **language** involving a single two-place **predicate symbol**. For instance, symmetry is defined by  $\forall x \forall y (Q(x, y) \rightarrow Q(y, x))$  in the sense that a first-order **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}| = W$  and  $Q^{\mathfrak{M}} = R$  satisfies the preceding formula iff  $R$  is symmetric. This suggests the following definition:

**Definition frd.1.** A class  $\mathcal{C}$  of frames is *first-order definable* if there is a **sentence**  $\varphi$  in the first-order language with a single two-place **predicate symbol**  $P$  such that  $\mathfrak{F} = \langle W, R \rangle \in \mathcal{C}$  iff  $\mathfrak{M} \models \varphi$  in the first-order **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}| = W$  and  $Q^{\mathfrak{M}} = R$ .

It turns out that the properties and modal **formulas** that define them considered so far are exceptional. Not every **formula** defines a first-order definable class of frames, and not every first-order definable class of frames is definable by a modal formula.

A counterexample to the first is given by the Löb formula:

$$\Box(\Box p \rightarrow p) \rightarrow \Box p. \quad (\text{W})$$

W defines the class of transitive and converse well-founded frames. A relation is well-founded if there is no infinite sequence  $w_1, w_2, \dots$  such that  $Rw_2w_1, Rw_3w_2, \dots$ . For instance, the relation  $<$  on  $\mathbb{N}$  is well-founded, whereas the relation  $<$  on  $\mathbb{Z}$  is not. A relation is converse well-founded iff its converse is well-founded. So converse well-founded relations are those where there is no infinite sequence  $w_1, w_2, \dots$  such that  $Rw_1w_2, Rw_2w_3, \dots$ .

There is, however, no first-order **formula** defining transitive converse well-founded relations. For suppose  $\mathfrak{M} \models \beta$  iff  $R = Q^{\mathfrak{M}}$  is transitive converse well-founded. Let  $\varphi_n$  be the **formula**

$$(Q(a_1, a_2) \wedge \dots \wedge Q(a_{n-1}, a_n))$$

Now consider the set of **formulas**

$$\Gamma = \{\beta, \varphi_1, \varphi_2, \dots\}.$$

Every finite subset of  $\Gamma$  is satisfiable: Let  $k$  be largest such that  $\varphi_k$  is in the subset,  $|\mathfrak{M}_k| = \{1, \dots, k\}$ ,  $a_i^{\mathfrak{M}_k} = i$ , and  $P^{\mathfrak{M}_k} = <$ . Since  $<$  on  $\{1, \dots, k\}$  is transitive and converse well-founded,  $\mathfrak{M}_k \models \beta$ .  $\mathfrak{M}_k \models \varphi_i$  by construction, for all  $i \leq k$ . By the Compactness Theorem for first-order logic,  $\Gamma$  is satisfiable in some **structure**  $\mathfrak{M}$ . By hypothesis, since  $\mathfrak{M} \models \beta$ , the relation  $Q^{\mathfrak{M}}$  is converse well-founded. But clearly,  $a_1^{\mathfrak{M}}, a_2^{\mathfrak{M}}, \dots$  would form an infinite sequence of the kind ruled out by converse well-foundedness.

A counterexample to the second claim is given by the property of universality: for every  $u$  and  $v$ ,  $Ruv$ . Universal frames are first-order definable by

the formula  $\forall x \forall y Q(x, y)$ . However, no modal formula is valid in all and only the universal frames. This is a consequence of a result that is independently interesting: the formulas valid in universal frames are exactly the same as those valid in reflexive, symmetric, and transitive frames. There are reflexive, symmetric, and transitive frames that are not universal, hence every formula valid in all universal frames is also valid in some non-universal frames.

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## Bibliography