We’ve seen that a number of properties of accessibility relations of frames can be defined by modal formulas. For instance, symmetry of frames can be defined by the formula $B, p \rightarrow \Box \Diamond p$. The conditions we’ve encountered so far can all be expressed by first-order formulas in a language involving a single two-place predicate symbol. For instance, symmetry is defined by $\forall x \forall y (Q(x, y) \rightarrow Q(y, x))$ in the sense that a first-order structure $\mathcal{M}$ with $|\mathcal{M}| = W$ and $Q^\mathcal{M} = R$ satisfies the preceding formula iff $R$ is symmetric. This suggests the following definition:

**Definition frd.1.** A class $\mathcal{F}$ of frames is first-order definable if there is a sentence $\varphi$ in the first-order language with a single two-place predicate symbol $Q$ such that $\mathcal{F} = \{ \mathcal{M} \in \mathcal{F} \mid \mathcal{M} \models \varphi \}$.

It turns out that the properties and modal formulas that define them considered so far are exceptional. Not every formula defines a first-order definable class of frames, and not every first-order definable class of frames is definable by a modal formula.

A counterexample to the first is given by the L"ob formula: $\Box(\Box p \rightarrow p) \rightarrow \Box p$. (W)

$W$ defines the class of transitive and converse well-founded frames. A relation is well-founded if there is no infinite sequence $w_1, w_2, \ldots$ such that $Rw_1w_2, Rw_3w_2, \ldots$. For instance, the relation $<$ on $\mathbb{N}$ is well-founded, whereas the relation $<$ on $\mathbb{Z}$ is not. A relation is converse well-founded iff its converse is well-founded. So converse well-founded relations are those where there is no infinite sequence $w_1, w_2, \ldots$ such that $Rw_1w_2, Rw_2w_3, \ldots$.

There is, however, no first-order formula defining transitive converse well-founded relations. For suppose $\mathcal{M} \models \beta$ iff $R = Q^\mathcal{M}$ is transitive converse well-founded. Let $\varphi_n$ be the formula

$$(Q(a_1, a_2) \land \cdots \land Q(a_{n-1}, a_n))$$

Now consider the set of formulas

$$\Gamma = \{ \beta, \varphi_1, \varphi_2, \ldots \}.$$  

Every finite subset of $\Gamma$ is satisfiable: Let $k$ be largest such that $\varphi_k$ is in the subset, $|\mathcal{M}_k| = \{1, \ldots, k\}$, $u^\mathcal{M}_k = i$, and $P^\mathcal{M}_k = <$. Since $<$ on $\{1, \ldots, k\}$ is transitive and converse well-founded, $\mathcal{M}_k \models \beta$. $\mathcal{M}_k \models \varphi_i$ by construction, for all $i \leq k$. By the Compactness Theorem for first-order logic, $\Gamma$ is satisfiable in some structure $\mathcal{M}$. By hypothesis, since $\mathcal{M} \models \beta$, the relation $Q^\mathcal{M}$ is converse well-founded. But clearly, $a_1^\mathcal{M}, a_2^\mathcal{M}, \ldots$ would form an infinite sequence of the kind ruled out by converse well-foundedness.
A counterexample to the second claim is given by the property of universality: for every \( u \) and \( v \), \( Ruv \). Universal frames are first-order definable by the formula \( \forall x \forall y Q(x, y) \). However, no modal formula is valid in all and only the universal frames. This is a consequence of a result that is independently interesting: the formulas valid in universal frames are exactly the same as those valid in reflexive, symmetric, and transitive frames. There are reflexive, symmetric, and transitive frames that are not universal, hence every formula valid in all universal frames is also valid in some non-universal frames.

**Photo Credits**

**Bibliography**