

frd.1 Equivalence Relations and S5

mod:frd:es5:
sec

The modal logic **S5** is characterized as the set of **formulas** valid on all universal frames, i.e., every world is accessible from every world, including itself. In such a scenario, \Box corresponds to necessity and \Diamond to possibility: $\Box\varphi$ is true if φ is true at *every* world, and $\Diamond\varphi$ is true if φ is true at *some* world. It turns out that **S5** can also be characterized as the **formulas** valid on all reflexive, symmetric, and transitive frames, i.e., on all *equivalence relations*.

Definition frd.1. A binary relation R on W is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation R on W is *universal* if and only if Ruv for all $u, v \in W$.

Since T, B, and 4 characterize the reflexive, symmetric, and transitive frames, the frames where the accessibility relation is an equivalence relation are exactly those in which all four **formulas** are valid. It turns out that the equivalence relations can also be characterized by other combinations of **formulas**, since the conditions with which we've defined equivalence relations are equivalent to combinations of other familiar conditions on R .

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prop:equivalences

Proposition frd.2. *The following are equivalent:*

1. R is an equivalence relation;
2. R is reflexive and euclidean;
3. R is serial, symmetric, and euclidean;
4. R is serial, symmetric, and transitive.

Proof. Exercise. □

Problem frd.1. Prove [Proposition frd.2](#) by showing:

1. If R is symmetric and transitive, it is euclidean.
2. If R is reflexive, it is serial.
3. If R is reflexive and euclidean, it is symmetric.
4. If R is symmetric and euclidean, it is transitive.
5. If R is serial, symmetric, and transitive, it is reflexive.

Explain why this suffices for the proof that the conditions are equivalent.

[Proposition frd.2](#) is the semantic counterpart to [??](#), in that it gives equivalent characterization of the modal logic of frames over which R is an equivalence (the logic traditionally referred to as **S5**).

What is the relationship between universal and equivalence relations? Although every universal relation is an equivalence relation, clearly not every equivalence relation is universal. However, the **formulas** valid on all universal relations are exactly the same as those valid on all equivalence relations.

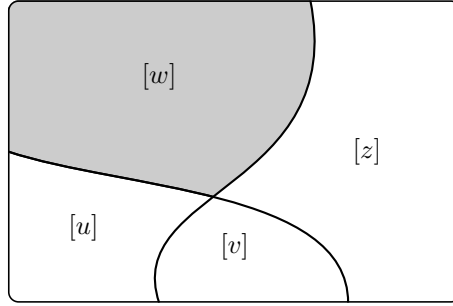


Figure 1: A partition of W in equivalence classes.

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fig:partition

Proposition frd.3. *Let R be an equivalence relation, and for each $w \in W$ define the equivalence class of w as the set $[w] = \{w' \in W : Rww'\}$. Then:*

1. $w \in [w]$;
2. R is universal on each equivalence class $[w]$;
3. The collection of equivalence classes partitions W into mutually exclusive and jointly exhaustive subsets.

Proposition frd.4. *A formula φ is valid in all frames $\mathfrak{F} = \langle W, R \rangle$ where R is an equivalence relation, if and only if it is valid in all frames $\mathfrak{F} = \langle W, R \rangle$ where R is universal. Hence, the logic of universal frames is just **S5**.*

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prop:S5=uni

Proof. It's immediate to verify that a universal relation R on W is an equivalence. Hence, if φ is valid in all frames where R is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose ψ is a formula that fails at a world w in a model $\mathfrak{M} = \langle W, R, V \rangle$ based on a frame $\langle W, R \rangle$, where R is an equivalence on W . So $\mathfrak{M}, w \not\models \psi$. Define a model $\mathfrak{M}' = \langle W', R', V' \rangle$ as follows:

1. $W' = [w]$;
2. R' is universal on W' ;
3. $V'(p) = V(p) \cap W'$.

(So the set W' of worlds in \mathfrak{M}' is represented by the shaded area in Figure 1.) It is easy to see that R and R' agree on W' . Then one can show by induction on formulas that for all $w' \in W'$: $\mathfrak{M}', w' \models \varphi$ if and only if $\mathfrak{M}, w' \models \varphi$ for each φ (this makes sense since $W' \subseteq W$). In particular, $\mathfrak{M}', w \not\models \psi$, and ψ fails in a model based on a universal frame. \square

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Bibliography