Filtrations allow us to establish the decidability of our systems of modal logic by showing that they have the finite model property, i.e., that any formula that is true (false) in a model is also true (false) in a finite model. Filtrations are defined relative to sets of formulas which are closed under subformulas.

**Definition fil.1.** A set $\Gamma$ of formulas is **closed under subformulas** if it contains every subformula of a formula in $\Gamma$. Further, $\Gamma$ is **modally closed** if it is closed under subformulas and moreover $\varphi \in \Gamma$ implies $\Box \varphi, \Diamond \varphi \in \Gamma$.

For instance, given a formula $\varphi$, the set of all its sub-formulas is closed under sub-formulas. When we’re defining a filtration of a model through the set of sub-formulas of $\varphi$, it will have the property we’re after: it makes $\varphi$ true (false) iff the original model does.

The set of worlds of a filtration of $\mathcal{M}$ through $\Gamma$ is defined as the set of all equivalence classes of the following equivalence relation.

**Definition fil.2.** Let $\mathcal{M} = \langle W, R, V \rangle$ and suppose $\Gamma$ is closed under sub-formulas. Define a relation $\equiv$ on $W$ to hold of any two worlds that make the same formulas from $\Gamma$ true, i.e.:

$$u \equiv v \quad \text{if and only if} \quad \forall \varphi \in \Gamma : \mathcal{M}, u \models \varphi \iff \mathcal{M}, v \models \varphi.$$  

The equivalence class $[w]_{\equiv}$ of a world $w$, or $[w]$ for short, is the set of all worlds $\equiv$-equivalent to $w$:

$$[w] = \{ v : v \equiv w \}.$$  

**Proposition fil.3.** Given $\mathcal{M}$ and $\Gamma$, $\equiv$ as defined above is an equivalence relation, i.e., it is reflexive, symmetric, and transitive.

**Proof.** The relation $\equiv$ is reflexive, since $w$ makes exactly the same formulas from $\Gamma$ true as itself. It is symmetric since if $u$ makes the same formulas from $\Gamma$ true as $v$, the same holds for $v$ and $u$. It is also transitive, since if $u$ makes the same formulas from $\Gamma$ true as $v$, and $v$ as $w$, then $u$ makes the same formulas from $\Gamma$ true as $w$.

The relation $\equiv$, like any equivalence relation, divides $W$ into **partitions**, i.e., subsets of $W$ which are pairwise disjoint, and together cover all of $W$. Every $w \in W$ is an element of one of the partitions, namely of $[w]$, since $w \equiv w$. So the partitions $[w]$ cover all of $W$. They are pairwise disjoint, for if $u \in [w]$ and $u \in [v]$, then $u \equiv w$ and $u \equiv v$, and by symmetry and transitivity, $w \equiv v$, and so $[w] = [v]$.

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**Bibliography**