fil.1 Preliminaries

mod:fil:pre:sec

Filtrations allow us to establish the decidability of our systems of modal logic by showing that they have the *finite model property*, i.e., that any formula that is true (false) in a model is also true (false) in a *finite* model. Filtrations are defined relative to sets of formulas which are closed under subformulas.

mod:fil:pre:defn:modallyclosed

Definition fil.1. A set Γ of formulas is *closed under subformulas* if it contains every subformula of a formula in Γ. Further, Γ is *modally closed* if it is closed under subformulas and moreover $\varphi \in \Gamma$ implies $\Box\varphi, \Diamond\varphi \in \Gamma$.

For instance, given a formula φ, the set of all its sub-formulas is closed under sub-formulas. When we're defining a filtration of a model through the set of sub-formulas of φ, it will have the property we're after: it makes φ true (false) iff the original model does.

The set of worlds of a filtration of $\mathfrak{M}$ through Γ is defined as the set of all equivalence classes of the following equivalence relation.

Definition fil.2. Let $\mathfrak{M} = \langle W, R, V\rangle$ and suppose Γ is closed under sub-formulas. Define a relation $\equiv$ on W to hold of any two worlds that make the same formulas from Γ true, i.e.:

$$u \equiv v \quad \text{if and only if} \quad \forall\varphi \in \Gamma : \mathfrak{M}, u \Vdash \varphi \Leftrightarrow \mathfrak{N}, v \Vdash \varphi.$$

The equivalence class $[w]_\equiv$ of a world w, or $[w]$ for short, is the set of all worlds $\equiv$-equivalent to w:

$$[w] = \{v : v \equiv w\}.$$

Proposition fil.3. *Given $\mathfrak{M}$ and Γ, $\equiv$ as defined above is an equivalence relation, i.e., it is reflexive, symmetric, and transitive.*

Proof. The relation $\equiv$ is reflexive, since w makes exactly the same formulas from Γ true as itself. It is symmetric since if u makes the same formulas from Γ true as v, the same holds for v and u. It is also transitive, since if u makes the same formulas from Γ true as v, and v as w, then u makes the same formulas from Γ true as w. □

The relation $\equiv$, like any equivalence relation, divides W into *partitions*, i.e., subsets of W which are pairwise disjoint, and together cover all of W. Every $w \in W$ is an element of one of the partitions, namely of $[w]$, since $w \equiv w$. So the partitions $[w]$ cover all of W. They are pairwise disjoint, for if $u \in [w]$ and $u \in [v]$, then $u \equiv w$ and $u \equiv v$, and by symmetry and transitivity, $w \equiv v$, and so $[w] = [v]$.

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Bibliography