

## fl.1 Preliminaries

mod:fil:pre:sec Filtrations allow us to establish the decidability of our systems of modal logic by showing that they have the *finite model property*, i.e., that any **formula** that is true (false) in a model is also true (false) in a *finite* model. Filtrations are defined relative to sets of **formulas** which are closed under subformulas.

mod:fil:pre: defn:modallyclosed **Definition fl.1.** A set  $\Gamma$  of **formulas** is *closed under subformulas* if it contains every subformula of a **formula** in  $\Gamma$ . Further,  $\Gamma$  is *modally closed* if it is closed under subformulas and moreover  $\varphi \in \Gamma$  implies  $\Box\varphi, \Diamond\varphi \in \Gamma$ .

For instance, given a **formula**  $\varphi$ , the set of all its sub-**formulas** is closed under sub-**formulas**. When we're defining a filtration of a model through the set of sub-**formulas** of  $\varphi$ , it will have the property we're after: it makes  $\varphi$  true (false) iff the original model does.

The set of worlds of a filtration of  $\mathfrak{M}$  through  $\Gamma$  is defined as the set of all equivalence classes of the following equivalence relation.

**Definition fl.2.** Let  $\mathfrak{M} = \langle W, R, V \rangle$  and suppose  $\Gamma$  is closed under sub-**formulas**. Define a relation  $\equiv$  on  $W$  to hold of any two worlds that make the same **formulas** from  $\Gamma$  true, i.e.:

$$u \equiv v \quad \text{if and only if} \quad \forall \varphi \in \Gamma : \mathfrak{M}, u \Vdash \varphi \Leftrightarrow \mathfrak{M}, v \Vdash \varphi.$$

The equivalence class  $[w]_{\equiv}$  of a world  $w$ , or  $[w]$  for short, is the set of all worlds  $\equiv$ -equivalent to  $w$ :

$$[w] = \{v : v \equiv w\}.$$

**Proposition fl.3.** *Given  $\mathfrak{M}$  and  $\Gamma$ ,  $\equiv$  as defined above is an equivalence relation, i.e., it is reflexive, symmetric, and transitive.*

*Proof.* The relation  $\equiv$  is reflexive, since  $w$  makes exactly the same **formulas** from  $\Gamma$  true as itself. It is symmetric since if  $u$  makes the same **formulas** from  $\Gamma$  true as  $v$ , the same holds for  $v$  and  $u$ . It is also transitive, since if  $u$  makes the same **formulas** from  $\Gamma$  true as  $v$ , and  $v$  as  $w$ , then  $u$  makes the same **formulas** from  $\Gamma$  true as  $w$ .  $\square$

The relation  $\equiv$ , like any equivalence relation, divides  $W$  into *partitions*, i.e., subsets of  $W$  which are pairwise disjoint, and together cover all of  $W$ . Every  $w \in W$  is an **element** of one of the partitions, namely of  $[w]$ , since  $w \equiv w$ . So the partitions  $[w]$  cover all of  $W$ . They are pairwise disjoint, for if  $u \in [w]$  and  $u \in [v]$ , then  $u \equiv w$  and  $u \equiv v$ , and by symmetry and transitivity,  $w \equiv v$ , and so  $[w] = [v]$ .

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## Bibliography