fil.1 Filtrations and Properties of Accessibility

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Definition fil.1. Let Γ be closed under subformulas and $\mathfrak{M} = \langle W, R, V\rangle$ a model. Then we can define conditions on pairs of worlds u, v as given in the table of Figure 1.

$C_1(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$;
$C_2(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then $\mathfrak{M}, u \models \varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$;
$C_3(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Diamond\varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$;
$C_4(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then $\mathfrak{M}, u \models \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Diamond\varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$;

Figure 1: Conditions on possible worlds for defining filtrations.

mod:fil:acc:fig:Cn-filtrations
mod:fil:acc:thm:more-filtrations

Theorem fil.2. *Let $\mathfrak{M} = \langle W, R, P\rangle$ be a model, Γ closed under subformulas. Let W^* and V^* be defined as in* **??**. *Then:*

1. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_2(u,v)$ then R^* is symmetric, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ is a filtration if $\mathfrak{M}$ is symmetric.*

2. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_3(u,v)$ then R^* is transitive, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ is a filtration if $\mathfrak{M}$ is transitive.*

3. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_2(u,v) \wedge C_3(u,v) \wedge C_4(u,v)$ then R^* is symmetric and transitive, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ is a filtration if $\mathfrak{M}$ is symmetric and transitive.*

4. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_3(u,v) \wedge C_4(u,v)$ then R^* is transitive and euclidean, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ is a filtration if $\mathfrak{M}$ is transitive and euclidean.*

Proof. 1. It's immediate that R^* is symmetric, since $C_1(u,v) \Leftrightarrow C_2(v,u)$ and $C_2(u,v) \Leftrightarrow C_1(v,u)$. So it's left to show that if $\mathfrak{M}$ is symmetric then $\mathfrak{M}^*$ is a filtration through Γ. By condition $C_1(u,v)$ we get that: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$, and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. So all we need is that Ruv implies $R^*[u][v]$.

So suppose Ruv, to show $R^*[u][v]$ we need $C_1(u,v) \wedge C_2(u,v)$. For C_1: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then also $\mathfrak{M}, v \models \varphi$ (since Ruv); and similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. For C_2: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then Ruv implies Rvu by symmetry, so that $\mathfrak{M}, u \models \varphi$; similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$ (since Rvu by symmetry).

2. Exercise.

3. Exercise.
4. Exercise.

□

Problem fil.1. Complete the proof of Theorem fil.2.

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Bibliography