

## fil.1 Filtrations are Finite

mod:fil:fin:  
sec We've defined filtrations for any set  $\Gamma$  that is closed under sub-formulas. Nothing in the definition itself guarantees that filtrations are finite. In fact, when  $\Gamma$  is infinite (e.g., is the set of all formulas), it may well be infinite. However, if  $\Gamma$  is finite (e.g., when it is the set of sub-formulas of a given formula  $\varphi$ ), so is any filtration through  $\Gamma$ .

mod:fil:fin:  
prop:filt-are-finite **Proposition fil.1.** *If  $\Gamma$  is finite then any filtration  $\mathfrak{M}^*$  of a model  $\mathfrak{M}$  through  $\Gamma$  is also finite.*

*Proof.* The size of  $W^*$  is the number of different classes  $[w]$  under the equivalence relation  $\equiv$ . Any two worlds  $u, v$  in such class—that is, any  $u$  and  $v$  such that  $u \equiv v$ —agree on all formulas  $\varphi$  in  $\Gamma$ ,  $\varphi \in \Gamma$  either  $\varphi$  is true at both  $u$  and  $v$ , or at neither. So each class  $[w]$  corresponds to subset of  $\Gamma$ , namely the set of all  $\varphi \in \Gamma$  such that  $\varphi$  is true at the worlds in  $[w]$ . No two different classes  $[u]$  and  $[v]$  correspond to the same subset of  $\Gamma$ . For if the set of formulas true at  $u$  and that of formulas true at  $v$  are the same, then  $u$  and  $v$  agree on all formulas in  $\Gamma$ , i.e.,  $u \equiv v$ . But then  $[u] = [v]$ . So, there is an injective function from  $W^*$  to  $\wp(\Gamma)$ , and hence  $|W^*| \leq |\wp(\Gamma)|$ . Hence if  $\Gamma$  contains  $n$  sentences, the cardinality of  $W^*$  is no greater than  $2^n$ .  $\square$

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## Bibliography