Chapter udf

Filtrations and Decidability

fil.1 Preliminaries

mod:fil:pre:sec

Filtrations allow us to establish the decidability of our systems of modal logic by showing that they have the *finite model property*, i.e., that any formula that is true (false) in a model is also true (false) in a *finite* model.

mod:fil:pre:def:modallyclosed

Definition fil.1. A set Γ of formulas is *closed under subformulas* if it contains every subformula of a formula in Γ. Further, Γ is *modally closed* if it is closed under subformulas and moreover $\varphi \in \Gamma$ implies $\Box\varphi, \Diamond\varphi \in \Gamma$.

Definition fil.2. Let $\mathfrak{M} = \langle W, R, V \rangle$ and suppose Γ is closed under subformulas. Define a relation $\equiv$ on W to hold of any two worlds that make true the same formulas from Γ, i.e.:

$$u \equiv v \quad \text{if and only if} \quad \forall \varphi \in \Gamma : \mathfrak{M}, u \models \varphi \Leftrightarrow \mathfrak{N}, v \models \varphi.$$

Clearly, $\equiv$ is an equivalence relation over W. Standardly, for any $w \in W$, the equivalence class of w is denoted by $[w]$.

fil.2 Filtrations

mod:fil:fil:sec

mod:fil:fil:def:filtration

Definition fil.3. Let Γ be closed under subformulas and $\mathfrak{M} = \langle W, R, V \rangle$. A *filtration of $\mathfrak{M}$ through Γ* is any model $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$, where:

1. $W^* = \{[w] : w \in W\}$;
2. mod:fil:fil:def:filtration-R For any $u, v \in W$:
   a) mod:fil:fil:def:filtration-R1 If Ruv then $R^*[u][v]$;
   b) mod:fil:fil:def:filtration-R2 If $R^*[u][v]$ then for any $\Box\varphi \in \Gamma$, if $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$;
   c) mod:fil:fil:def:filtration-R3 If $R^*[u][v]$ then for any $\Diamond\varphi \in \Gamma$, if $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$.

3. $V^*(p) = \{[u] : u \in V(p)\}$.

Theorem fil.4. *If $\mathfrak{M}^*$ is a filtration of $\mathfrak{M}$ through Γ, then for every $\varphi \in \Gamma$ and $w \in W$, we have $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M}^*, [w] \models \varphi$.* mod:fil:fil:thm:filtrations

Proof. By induction on φ, using the fact that Γ is closed under subformulas. For φ atomic, say p: the left-to-right direction is immediate, as $\mathfrak{M}, w \models p$ only if $w \in V(p)$, which implies $[w] \in V^*(p)$, i.e., $\mathfrak{M}^*, [w] \models p$. Conversely, suppose $\mathfrak{M}^*, [w] \models p$, i.e., $[w] \in V^*(p)$; then $w \equiv w' \in V(p)$, and since $p \in \Gamma$, also $w \in V(p)$, so that $\mathfrak{M}, w \models p$. The cases for the Boolean connectives follow immediately from the inductive hypothesis and closure of Γ under subformulas.

So we do the case for $\Box\varphi \in \Gamma$. Suppose $\mathfrak{M}, u \models \Box\varphi$; to show that $\mathfrak{M}^*, [u] \models \varphi$, let v be such that $R^*[u][v]$. From Definition fil.3(2b), we have that $\mathfrak{M}, v \models \varphi$, and by inductive hypothesis $\mathfrak{M}^*, [v] \models \varphi$. Since v was arbitrary, $\mathfrak{M}^*, [u] \models \Box\varphi$ follows. Conversely, suppose $\mathfrak{M}^*, [u] \models \Box\varphi$ and let v be arbitrary such that Ruv. From Definition fil.3(2a), we have $R^*[u][v]$, so that $\mathfrak{M}^*, [v] \models \varphi$; by inductive hypothesis $\mathfrak{M}, v \models \varphi$, and since v was arbitrary, $\mathfrak{M}, u \models \Box\varphi$. □

Corollary fil.5. *Let Γ be closed under subformulas. Then:*

1. *If $\mathfrak{M}^*$ is a filtration of $\mathfrak{M}$ through Γ then for any $\varphi \in \Gamma$: $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{M}^* \models \varphi$.*

2. *If $\mathcal{C}$ is a class of models and $\Gamma(\mathcal{C})$ is the class of Γ-filtrations of models in $\mathcal{C}$, then any formula $\varphi \in \Gamma$ is valid in $\mathcal{C}$ if and only if it is valid in $\Gamma(\mathcal{C})$.*

fil.3 Examples of Filtrations

We have not yet shown that there are any filtrations. But indeed, for any model $\mathfrak{M}$, there are many filtrations of $\mathfrak{M}$ through Γ. We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as Definition fil.3 stipulates directly what W^* and V^* should be like). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible. mod:fil:exf:sec

Definition fil.6. Where Γ is closed under subformulas, the *finest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting:

$$R^*[u][v] \quad \text{if and only if} \quad \exists u' \in [u]\, \exists v' \in [v] : Ru'v'.$$

Proposition fil.7. *The finest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Proof. We need to check that R^*, so defined, satisfies Definition fil.3(2). We check the three conditions in turn.

If Ruv then by reflexivity of $\equiv$, also $R^*[u][v]$, so (2a) is satisfied.

For (2b), suppose $\Box\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, u \models \Box\varphi$. By definition of R^*, there are $u' \equiv u$ an $v' \equiv v$ such that $Ru'v'$. Since u and u' agree on Γ, also

$\mathfrak{M}, u' \models \Box\varphi$, so that $\mathfrak{M}, v' \models \varphi$. By closure of Γ, v and v' agree on φ, so $\mathfrak{M}, v \models \varphi$, as desired.

To verify (2c), suppose $\Diamond\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, v \models \varphi$. Arguing similarly to the previous case, $\mathfrak{M}, u \models \Diamond\varphi$. □

Definition fil.8. Where Γ is closed under subformulas, the *coarsest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting $R^*[u][v]$ if and only if *both* of the following conditions are met:

1. If $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$;
2. If $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$.

Proposition fil.9. *The coarsest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Proof. Given the definition of R^*, the only condition that is left to verify is the implication from Ruv to $R^*[u][v]$. Assuming Ruv, suppose $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$; then obviously $\mathfrak{M}, v \models \varphi$. Similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. □

fil.4 Filtrations are Finite

mod:fil:fin:sec

mod:fil:fin:prop:filt-are-finite **Proposition fil.10.** *If Γ is finite then any filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ through Γ is also finite.*

Proof. If $u \equiv v$ then, by Theorem fil.4, the set of $\varphi \in \Gamma$ that are true at u is the same as the set of $\varphi \in \Gamma$ that are true at v. So to each $[u] \in W^*$ we can assign a *distinct* subset of Γ. Hence if Γ contains n sentences the cardinality of W^* is no greater than 2^n. □

fil.5 S5 has the Finite Model Property

mod:fil:fmp:sec

Definition fil.11. A system Σ of modal logic is said to have the *finite model property* if whenever a formula φ is true at a world in a model of Σ then φ is true at a world in a *finite* model of Σ.

mod:fil:fmp:prop:univ-fin **Proposition fil.12.** *Let $\mathcal{U}$ be the class of universal models (see* **??***) and $\mathcal{U}_{\text{Fin}}$ the class of all finite universal models. Then any formula φ is valid in $\mathcal{U}$ if and only if it is valid in $\mathcal{U}_{\text{Fin}}$.*

Proof. Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to left direction, suppose that φ is false at some world w in a universal model $\mathfrak{M}$. Let Γ contain φ as well as all of its subformulas; clearly Γ is finite. Take a filtration $\mathfrak{M}^*$ of $\mathfrak{M}$; then $\mathfrak{M}^*$ is finite by

Proposition fil.10, and by Theorem fil.4, φ is false at $[w]$ in $\mathfrak{M}^*$. It remains to observe that $\mathfrak{M}^*$ is also universal: given u and v, by hypothesis Ruv and by Definition Definition fil.3(2), also $R^*[u][v]$. □

Corollary fil.13. **S5** *has the finite model property.* *mod:fil:fmp: cor:S5fmp*

Proof. By **??** and Proposition fil.12, if φ is true at a world in some reflexive and euclidean model then it is true at a world in a finite universal model (universal models are obviously reflexive and euclidean). □

Problem fil.1. Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

Problem fil.2. Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

fil.6 S5 is Decidable

The finite model property gives us an easy way to show that systems of modal logic given by schemas are *decidable* (i.e., that there is a computable procedure to determine whether a formulas is derivable in the system or not). mod:fil:dec: sec

Theorem fil.14. **S5** *is decidable.*

Proof. Let φ be given, and suppose the propositional variables occurring in φ are among $p_1, \ldots, p_k$. Since for each n there are only finitely many models with n worlds assigning a value to $p_1, \ldots, p_k$, we can enumerate, *in parallel*, all the theorems of **S5** by generating proofs in some systematic way; and all the models containing $1, 2, \ldots$ worlds and checking whether φ fails at a world in some such model. Eventually one of the two parallel processes will give an answer, as by **??** and Corollary fil.13, either φ is derivable or it fails in a finite universal model. □

The above proof works for **S5** because filtrations of universal models are automatically universal. The same holds for reflexivity and seriality, but more work is needed for other properties.

Problem fil.3. Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

Problem fil.4. Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

fil.7 Filtrations and Properties of Accessibility

mod:fil:acc:sec

Definition fil.15. Let Γ be closed under subformulas and $\mathfrak{M} = \langle W, R, V \rangle$ a model. Then we can define conditions on pairs of worlds u, v as given in the table of Figure fil.1.

$C_1(u, v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$;
$C_2(u, v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then $\mathfrak{M}, u \models \varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$;
$C_3(u, v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Diamond\varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$;
$C_4(u, v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then $\mathfrak{M}, u \models \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Diamond\varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$;

Figure fil.1: Conditions on possible worlds for defining filtrations.

mod:fil:acc:fig:Cn-filtrations mod:fil:acc:thm:more-filtrations

Theorem fil.16. *Let $\mathfrak{M} = \langle W, R, P \rangle$ be a model, Γ closed under subformulas. Let W^* and V^* be defined as in Definition fil.3. Then:*

1. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_2(u, v)$ then R^* is symmetric, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a filtration if $\mathfrak{M}$ is symmetric.*

2. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_3(u, v)$ then R^* is transitive, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a filtration if $\mathfrak{M}$ is transitive.*

3. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_2(u, v) \wedge C_3(u, v) \wedge C_4(u, v)$ then R^* is symmetric and transitive, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a filtration if $\mathfrak{M}$ is symmetric and transitive.*

4. *If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \wedge C_3(u, v) \wedge C_4(u, v)$ then R^* is transitive and euclidean, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a filtration if $\mathfrak{M}$ is transitive and euclidean.*

Proof. 1. It's immediate that R^* is symmetric, since $C_1(u, v) \Leftrightarrow C_2(v, u)$ and $C_2(u, v) \Leftrightarrow C_1(v, u)$. So it's left to show that if $\mathfrak{M}$ is symmetric then $\mathfrak{M}^*$ is a filtration through Γ. By condition $C_1(u, v)$ we get that: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$, and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. So all we need is that Ruv implies $R^*[u][v]$.

So suppose Ruv, to show $R^*[u][v]$ we need $C_1(u, v) \wedge C_2(u, v)$. For C_1: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then also $\mathfrak{M}, v \models \varphi$ (since Ruv); and similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. For C_2: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \models \Box\varphi$ then Ruv implies Rvu by symmetry, so that $\mathfrak{M}, u \models \varphi$; similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \models \varphi$ then $\mathfrak{M}, v \models \Diamond\varphi$ (since Rvu by symmetry).

2. Exercise.

3. Exercise.

4. Exercise.

□

Problem fil.5. Complete the proof of Theorem fil.16.

fil.8 Filtrations of Euclidean Models

The approach of section fil.7 does not work in the case of models that are euclidean or serial and euclidean. Consider the model at the top of Figure fil.2, which is both euclidean and serial. Let $\Gamma = \{p, \Box p\}$. When taking a filtration through Γ, then $[w_1] = [w_3]$ since w_1 and w_3 are the only worlds that agree on Γ. Any filtration will also have the arrow inherited from $\mathfrak{M}$, as depicted in Figure fil.3. But we cannot add arrows to that model in order to make it euclidean, for then there would be a double arrow between w_2 and w_4, and hence also between w_2 and w_5. But $\Box p$ is true at w_2 while p is false at w_5. mod:fil:euc:sec

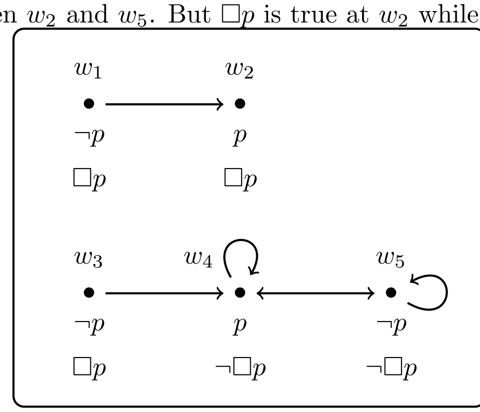

Figure fil.2: A serial and euclidean model. mod:fil:euc:fig:ser-eucl

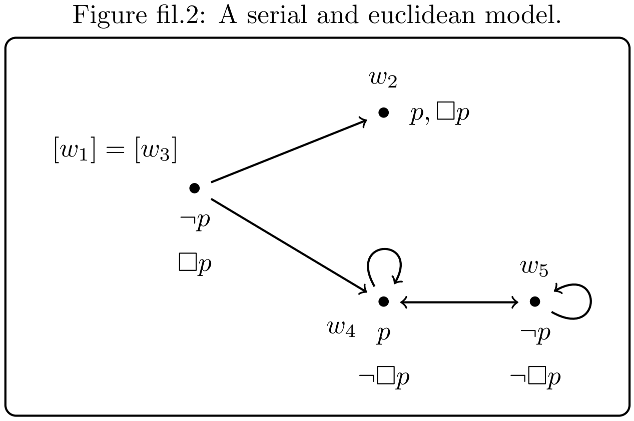

Figure fil.3: The filtration of the model in Figure fil.2. mod:fil:euc:fig:ser-eucl2

In particular, it is not enough to consider filtrations through arbitrary Γ's closed under subsentences. Instead we need to consider sets Γ that are *modally closed* (see Definition fil.1). Such sets of sentences are infinite, and therefore do not lead immediately to the decidability of the corresponding system.

Theorem fil.17. *Let Γ be modally closed and $\mathfrak{M} = \langle W, R, V \rangle$. If $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a coarsest filtration of $\mathfrak{M}$, then $\mathfrak{M}^*$ is symmetric, transitive or euclidean if $\mathfrak{M}$ is symmetric, transitive, or euclidean, respectively.*

Proof. The proof of transitivity uses the validity of both 4 and $4_\Diamond$ in all transitive models, and likewise euclideanness uses the fact that both 5 and $5_\Diamond$ are valid in all euclidean models, and the proof of symmetry likewise uses both B and $\mathrm{B}_\Diamond$.

If $\mathfrak{M}^*$ is a coarsest filtration, then by definition $R^*[u][v]$ holds if and only if $C_1(u, v)$. For transitivity, suppose $C_1(u, v)$ and $C_1(v, w)$: to show $C_1(u, w)$ suppose $\mathfrak{M}, u \models \Box\varphi$; then $\mathfrak{M}, u \models \Box\Box\varphi$; since $\Box\Box\varphi \in \Gamma$ by closure, also by $C_1(u, v)$, $\mathfrak{M}, v \models \Box\varphi$ and by $C_1(v, w)$, also $\mathfrak{M}, w \models \varphi$. The case for $\Diamond\varphi$ is similar. $\Box$

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Bibliography