fil.1 Filtrations

Rather than define “the” filtration of \( \mathcal{M} \) through \( \Gamma \), we define when a model \( \mathcal{M}^* \) counts as a filtration of \( \mathcal{M} \). All filtrations have the same set of worlds \( W^* \) and the same valuation \( V^* \). But different filtrations may have different accessibility relations \( R^* \). To count as a filtration, \( R^* \) has to satisfy a number of conditions, however. These conditions are exactly what we’ll require to prove the main result, namely that \( \mathcal{M}, w \models \varphi \) if \( \mathcal{M}^*, [w] \models \varphi \), provided \( \varphi \in \Gamma \).

Definition fil.1. Let \( \Gamma \) be closed under subformulas and \( \mathcal{M} = \langle W, R, V \rangle \). A filtration of \( \mathcal{M} \) through \( \Gamma \) is any model \( \mathcal{M}^* = \langle W^*, R^*, V^* \rangle \), where:

1. \( W^* = \{ [w] : w \in W \} \);
2. For any \( u, v \in W \):
   a) If \( Ruw \) then \( R^*[u][v] \);
   b) If \( R^*[u][v] \) then for any \( \Box \varphi \in \Gamma \), if \( \mathcal{M}, u \models \Box \varphi \) then \( \mathcal{M}, v \models \varphi \);
   c) If \( R^*[u][v] \) then for any \( \Diamond \varphi \in \Gamma \), if \( \mathcal{M}, v \models \varphi \) then \( \mathcal{M}, u \models \Diamond \varphi \);
3. \( V^*(p) = \{ [w] : u \in V(p) \} \).

It’s worthwhile thinking about what \( V^*(p) \) is: the set consisting of the equivalence classes \( [w] \) of all worlds \( w \) where \( p \) is true in \( \mathcal{M} \). On the one hand, if \( w \in V(p) \), then \( [w] \in V^*(p) \) by that definition. However, it is not necessarily the case that if \( [w] \in V^*(p) \), then \( w \in V(p) \). If \( [w] \in V^*(p) \) we are only guaranteed that \( [w] = [u] \) for some \( u \in V(p) \). Of course, \( [w] = [u] \) means that \( w \equiv u \). So, when \( [w] \in V^*(p) \) we can (only) conclude that \( w \equiv u \) for some \( u \in V(p) \).

Theorem fil.2. If \( \mathcal{M}^* \) is a filtration of \( \mathcal{M} \) through \( \Gamma \), then for every \( \varphi \in \Gamma \) and \( w \in W \), we have \( \mathcal{M}, w \models \varphi \) if and only if \( \mathcal{M}^*, [w] \models \varphi \).

Proof. By induction on \( \varphi \), using the fact that \( \Gamma \) is closed under subformulas. Since \( \varphi \in \Gamma \) and \( \Gamma \) is closed under sub-formulas, all sub-formulas of \( \varphi \) are also \( \in \Gamma \). Hence in each inductive step, the induction hypothesis applies to the sub-formulas of \( \varphi \).

1. \( \varphi \equiv \bot \): Neither \( \mathcal{M}, w \models \varphi \) nor \( \mathcal{M}^*, [w] \models \varphi \).
2. \( \varphi \equiv \top \): Both \( \mathcal{M}, w \models \varphi \) and \( \mathcal{M}^*, [w] \models \varphi \).
3. \( \varphi \equiv p \): The left-to-right direction is immediate, as \( \mathcal{M}, w \models \varphi \) only if \( w \in V(p) \), which implies \( [w] \in V^*(p) \), i.e., \( \mathcal{M}^*, [w] \models \varphi \). Conversely, suppose \( \mathcal{M}^*, [w] \models \varphi \), i.e., \( [w] \in V^*(p) \). Then for some \( v \in V(p) \), \( w \equiv v \). Of course then also \( \mathcal{M}, v \models p \). Since \( w \equiv v \), \( w \) and \( v \) make the same formulas from \( \Gamma \) true. Since by assumption \( p \in \Gamma \) and \( \mathcal{M}, v \models p \), \( \mathcal{M}, w \models \varphi \).
4. \( \varphi \equiv \neg \psi \): \( \mathcal{M}, w \models \varphi \) iff \( \mathcal{M}, w \not\models \psi \). By induction hypothesis, \( \mathcal{M}, w \not\models \psi \) iff \( \mathcal{M}^*, [w] \not\models \psi \). Finally, \( \mathcal{M}^*, [w] \not\models \psi \) iff \( \mathcal{M}^*, [w] \not\models \varphi \).
5. \( \varphi \equiv (\psi \land \chi) \): \( M, w \models \varphi \iff M, w \models \psi \) and \( M, w \models \chi \). By induction hypothesis, \( M, w \models \psi \) iff \( M^*, [w] \models \psi \), and \( M, w \models \chi \) iff \( M^*, [w] \models \chi \). And \( M^*, [w] \models \psi \) and \( M^*, [w] \models \chi \).

6. \( \varphi \equiv (\psi \lor \chi) \): \( M, w \models \varphi \iff M, w \not\models \psi \) or \( M, w \models \chi \). By induction hypothesis, \( M, w \models \psi \) iff \( M^*, [w] \models \psi \), and \( M, w \models \chi \) iff \( M^*, [w] \models \chi \). And \( M^*, [w] \models \psi \) or \( M^*, [w] \models \chi \).

7. \( \varphi \equiv (\psi \rightarrow \chi) \): \( M, w \models \varphi \iff M, w \not\models \psi \) and \( M, w \models \chi \), or \( M, w \not\models \psi \) and \( M, w \not\models \chi \). By induction hypothesis, \( M, w \not\models \psi \) iff \( M^*, [w] \not\models \psi \), and \( M, w \models \chi \) iff \( M^*, [w] \models \chi \). And \( M^*, [w] \not\models \psi \) and \( M^*, [w] \not\models \chi \).

8. \( \varphi \equiv (\psi \leftrightarrow \chi) \): \( M, w \models \varphi \iff M, w \models \psi \) and \( M, w \models \chi \), or \( M, w \not\models \psi \) and \( M, w \not\models \chi \). By induction hypothesis, \( M, w \models \psi \) iff \( M^*, [w] \models \psi \), and \( M, w \models \chi \) iff \( M^*, [w] \models \chi \). And \( M^*, [w] \models \psi \) or \( M^*, [w] \not\models \chi \).

9. \( \varphi \equiv \square \psi \): Suppose \( M, w \models \varphi \); to show that \( M^*, [w] \models \varphi \), let \( v \) be such that \( R^*[w][v] \). From Definition fil.1(2a), we have that \( M, v \models \psi \), and by inductive hypothesis \( M^*, [v] \models \psi \). Since \( v \) was arbitrary, \( M^*, [w] \models \varphi \) follows.

Conversely, suppose \( M^*, [w] \models \varphi \) and let \( v \) be arbitrary such that \( R^*[w][v] \). From Definition fil.1(2a), we have \( R^*[w][v] \), so that \( M^*, [v] \models \psi \); by inductive hypothesis \( M, v \models \psi \), and since \( v \) was arbitrary, \( M, w \models \varphi \).

10. \( \varphi \equiv 
    \): Suppose \( M, w \models \varphi \). Then for some \( v \in W \), \( R^*[w][v] \), \( M, v \models \psi \). By inductive hypothesis \( M^*, [v] \models \psi \), and by Definition fil.1(2a), we have \( R^*[w][v] \). Thus, \( M^*, [w] \models \varphi \).

Now suppose \( M^*, [w] \models \varphi \). Then for some \( [v] \in W^* \) with \( R^*[w][v] \), \( M^*, [v] \models \psi \). By inductive hypothesis \( M, v \models \psi \). By Definition fil.1(2c), we have that \( M, w \models \varphi \).

Problem fil.1. Complete the proof of Theorem fil.2

What holds for truth at worlds in a model also holds for truth in a model and validity in a class of models.

Corollary fil.3. Let \( \Gamma \) be closed under subformulas. Then:

1. If \( M^* \) is a filtration of \( M \) through \( \Gamma \) then for any \( \varphi \in \Gamma \): \( M \models \varphi \) if and only if \( M^* \models \varphi \).

2. If \( C \) is a class of models and \( \Gamma(C) \) is the class of \( \Gamma \)-filtrations of models in \( C \), then any formula \( \varphi \in \Gamma \) is valid in \( C \) if and only if it is valid in \( \Gamma(C) \).
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