### Filtrations

Rather than define “the” filtration of \( \mathcal{M} \) through \( \Gamma \), we define when a model \( \mathcal{M}^* \) counts as a filtration of \( \mathcal{M} \). All filtrations have the same set of worlds \( W^* \) and the same valuation \( V^* \). But different filtrations may have different accessibility relations \( R^* \). To count as a filtration, \( R^* \) has to satisfy a number of conditions, however. These conditions are exactly what we’ll require to prove the main result, namely that \( \mathcal{M}, w \vDash \varphi \) if \( \mathcal{M}^*, [w] \vDash \varphi \), provided \( \varphi \in \Gamma \).

**Definition fil.1.** Let \( \Gamma \) be closed under subformulas and \( \mathcal{M} = \langle W, R, V \rangle \). A filtration of \( \mathcal{M} \) through \( \Gamma \) is any model \( \mathcal{M}^* = \langle W^*, R^*, V^* \rangle \), where:

1. \( W^* = \{ [w] : w \in W \} \);
2. For any \( u, v \in W \):
   a) If \( Ruv \) then \( R^*[u][v] \);
   b) If \( R^*[u][v] \) then for any \( \Box \varphi \in \Gamma \), if \( \mathcal{M}, u \vDash \Box \varphi \) then \( \mathcal{M}, v \vDash \varphi \);
   c) If \( R^*[u][v] \) then for any \( \Diamond \varphi \in \Gamma \), if \( \mathcal{M}, v \vDash \varphi \) then \( \mathcal{M}, u \vDash \Diamond \varphi \).
3. \( V^*(p) = \{ [u] : u \in V(p) \} \).

It’s worthwhile thinking about what \( V^*(p) \) is: the set consisting of the equivalence classes \( [w] \) of all worlds \( w \) where \( p \) is true in \( \mathcal{M} \). On the one hand, if \( w \in V(p) \), then \( [w] \in V^*(p) \) by that definition. However, it is not necessarily the case that if \( [w] \in V^*(p) \), then \( w \in V(p) \). If \( [w] \in V^*(p) \) we are only guaranteed that \( [w] = [u] \) for some \( u \in V(p) \). Of course, \( [w] = [u] \) means that \( w \equiv u \). So, when \( [w] \in V^*(p) \) we can (only) conclude that \( w \equiv u \) for some \( u \in V(p) \).

**Theorem fil.2.** If \( \mathcal{M}^* \) is a filtration of \( \mathcal{M} \) through \( \Gamma \), then for every \( \varphi \in \Gamma \) and \( w \in W \), we have \( \mathcal{M}, w \vDash \varphi \) if and only if \( \mathcal{M}^*, [w] \vDash \varphi \).

*Proof.* By induction on \( \varphi \), using the fact that \( \Gamma \) is closed under subformulas. Since \( \varphi \in \Gamma \) and \( \Gamma \) is closed under sub-formulas, all sub-formulas of \( \varphi \) are also \( \in \Gamma \). Hence in each inductive step, the induction hypothesis applies to the sub-formulas of \( \varphi \).

1. \( \varphi \equiv \bot \): Neither \( \mathcal{M}, w \vDash \varphi \) nor \( \mathcal{M}^*, [w] \vDash \varphi \).
2. \( \varphi \equiv \top \): Both \( \mathcal{M}, w \vDash \varphi \) and \( \mathcal{M}^*, [w] \vDash \varphi \).
3. \( \varphi \equiv p \): The left-to-right direction is immediate, as \( \mathcal{M}, w \vDash \varphi \) only if \( w \in V(p) \), which implies \( [w] \in V^*(p) \), i.e., \( \mathcal{M}^*, [w] \vDash \varphi \). Conversely, suppose \( \mathcal{M}^*, [w] \vDash \varphi \), i.e., \( [w] \in V^*(p) \). Then for some \( v \in V(p) \), \( w \equiv v \). Of course then also \( \mathcal{M}, v \vDash p \). Since \( w \equiv v \), \( w \) and \( v \) make the same formulas from \( \Gamma \) true. Since by assumption \( p \in \Gamma \) and \( \mathcal{M}, v \vDash p \), \( \mathcal{M}, w \vDash \varphi \).
4. \( \varphi \equiv \neg \psi \): \( \mathcal{M}, w \vDash \varphi \) iff \( \mathcal{M}, w \not\vDash \psi \). By induction hypothesis, \( \mathcal{M}, w \not\vDash \psi \) iff \( \mathcal{M}^*, [w] \not\vDash \psi \). Finally, \( \mathcal{M}^*, [w] \not\vDash \psi \) iff \( \mathcal{M}^*, [w] \vDash \varphi \).
5. \( \varphi \equiv (\psi \land \chi) \): \( \mathcal{M}, w \models \varphi \) iff \( \mathcal{M}, w \models \psi \) and \( \mathcal{M}, w \models \chi \). By induction hypothesis, \( \mathcal{M}, w \models \psi \) iff \( \mathcal{M}^*, [w] \models \psi \), and \( \mathcal{M}, w \models \chi \) iff \( \mathcal{M}^*, [w] \models \chi \). And \( \mathcal{M}^*, [w] \models \varphi \) iff \( \mathcal{M}^*, [w] \models \psi \) and \( \mathcal{M}^*, [w] \models \chi \).

6. \( \varphi \equiv (\psi \lor \chi) \): \( \mathcal{M}, w \models \varphi \) iff \( \mathcal{M}, w \not\models \psi \) or \( \mathcal{M}, w \models \chi \). By induction hypothesis, \( \mathcal{M}, w \models \psi \) iff \( \mathcal{M}^*, [w] \models \psi \), and \( \mathcal{M}, w \models \chi \) iff \( \mathcal{M}^*, [w] \models \chi \). And \( \mathcal{M}^*, [w] \models \varphi \) iff \( \mathcal{M}^*, [w] \models \psi \) or \( \mathcal{M}^*, [w] \models \chi \).

7. \( \varphi \equiv (\psi \rightarrow \chi) \): \( \mathcal{M}, w \models \varphi \) iff \( \mathcal{M}, w \not\models \psi \) and \( \mathcal{M}, w \models \chi \), or \( \mathcal{M}, w \not\models \psi \) and \( \mathcal{M}, w \not\models \chi \). By induction hypothesis, \( \mathcal{M}, w \models \psi \) iff \( \mathcal{M}^*, [w] \models \psi \), and \( \mathcal{M}, w \models \chi \) iff \( \mathcal{M}^*, [w] \models \chi \). And \( \mathcal{M}^*, [w] \models \varphi \) iff \( \mathcal{M}^*, [w] \not\models \psi \) and \( \mathcal{M}^*, [w] \not\models \chi \).

8. \( \varphi \equiv (\psi \leftrightarrow \chi) \): \( \mathcal{M}, w \models \varphi \) iff \( \mathcal{M}, w \models \psi \) and \( \mathcal{M}, w \models \chi \), or \( \mathcal{M}, w \not\models \psi \) and \( \mathcal{M}, w \not\models \chi \). By induction hypothesis, \( \mathcal{M}, w \models \psi \) iff \( \mathcal{M}^*, [w] \models \psi \), and \( \mathcal{M}, w \models \chi \) iff \( \mathcal{M}^*, [w] \models \chi \). And \( \mathcal{M}^*, [w] \models \varphi \) iff \( \mathcal{M}^*, [w] \not\models \psi \) and \( \mathcal{M}^*, [w] \not\models \chi \).

9. \( \varphi \equiv \square \psi \): Suppose \( \mathcal{M}, w \models \varphi \); to show that \( \mathcal{M}^*, [w] \models \varphi \), let \( v \) be such that \( R^*[w][v] \). From Definition fil.1(2b), we have that \( \mathcal{M}, v \models \psi \), and by inductive hypothesis \( \mathcal{M}^*, [v] \models \psi \). Since \( v \) was arbitrary, \( \mathcal{M}^*, [w] \models \varphi \) follows.

Conversely, suppose \( \mathcal{M}^*, [w] \models \varphi \) and let \( v \) be arbitrary such that \( R^*[w][v] \). From Definition fil.1(2a), we have \( R^*[w][v] \), so that \( \mathcal{M}^*, [v] \models \psi \); by inductive hypothesis \( \mathcal{M}, v \models \psi \), and since \( v \) was arbitrary, \( \mathcal{M}, w \models \varphi \).

10. \( \varphi \equiv \lozenge \psi \): Suppose \( \mathcal{M}, w \models \varphi \). Then for some \( v \in W \), \( R^*[w][v] \) and \( \mathcal{M}, v \models \psi \). By inductive hypothesis \( \mathcal{M}^*, [v] \models \psi \), and by Definition fil.1(2a), we have \( R^*[w][v] \). Thus, \( \mathcal{M}^*, [w] \models \varphi \).

Now suppose \( \mathcal{M}^*, [w] \models \varphi \). Then for some \( [v] \in W^* \) with \( R^*[w][v] \), \( \mathcal{M}^*, [v] \models \psi \). By inductive hypothesis \( \mathcal{M}, v \models \psi \). By Definition fil.1(2c), we have that \( \mathcal{M}, w \models \varphi \). \( \square \)

Problem fil.1. Complete the proof of Theorem fil.2

What holds for truth at worlds in a model also holds for truth in a model and validity in a class of models.

Corollary fil.3. Let \( \Gamma \) be closed under subformulas. Then:

1. If \( \mathcal{M}^* \) is a filtration of \( \mathcal{M} \) through \( \Gamma \) then for any \( \varphi \in \Gamma \): \( \mathcal{M} \models \varphi \) if and only if \( \mathcal{M}^* \models \varphi \).

2. If \( C \) is a class of models and \( \Gamma(C) \) is the class of \( \Gamma \)-filtrations of models in \( C \), then any formula \( \varphi \in \Gamma \) is valid in \( C \) if and only if it is valid in \( \Gamma(C) \).

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Bibliography