fil.1 Filtrations

mod:fil:fil:sec

mod:fil:fil:def:filtration **Definition fil.1.** Let Γ be closed under subformulas and $\mathfrak{M} = \langle W, R, V\rangle$. A *filtration of* $\mathfrak{M}$ *through* Γ is any model $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$, where:

1. $W^* = \{[w] : w \in W\}$;
2. mod:fil:fil:def:filtration-R For any $u, v \in W$:
   a) mod:fil:fil:def:filtration-R1 If Ruv then $R^*[u][v]$;
   b) mod:fil:fil:def:filtration-R2 If $R^*[u][v]$ then for any $\Box\varphi \in \Gamma$, if $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$;
   c) mod:fil:fil:def:filtration-R3 If $R^*[u][v]$ then for any $\Diamond\varphi \in \Gamma$, if $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$.
3. $V^*(p) = \{[u] : u \in V(p)\}$.

mod:fil:fil:thm:filtrations **Theorem fil.2.** *If $\mathfrak{M}^*$ is a filtration of $\mathfrak{M}$ through Γ, then for every $\varphi \in \Gamma$ and $w \in W$, we have $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M}^*, [w] \models \varphi$.*

Proof. By induction on φ, using the fact that Γ is closed under subformulas. For φ atomic, say p: the left-to-right direction is immediate, as $\mathfrak{M}, w \models p$ only if $w \in V(p)$, which implies $[w] \in V^*(p)$, i.e., $\mathfrak{M}^*, [w] \models p$. Conversely, suppose $\mathfrak{M}^*, [w] \models p$, i.e., $[w] \in V^*(p)$; then $w \equiv w' \in V(p)$, and since $p \in \Gamma$, also $w \in V(p)$, so that $\mathfrak{M}, w \models p$. The cases for the Boolean connectives follow immediately from the inductive hypothesis and closure of Γ under subformulas.

So we do the case for $\Box\varphi \in \Gamma$. Suppose $\mathfrak{M}, u \models \Box\varphi$; to show that $\mathfrak{M}^*, [u] \models \varphi$, let v be such that $R^*[u][v]$. From Definition fil.1(2b), we have that $\mathfrak{M}, v \models \varphi$, and by inductive hypothesis $\mathfrak{M}^*, [v] \models \varphi$. Since v was arbitrary, $\mathfrak{M}^*, [u] \models \Box\varphi$ follows. Conversely, suppose $\mathfrak{M}^*, [u] \models \Box\varphi$ and let v be arbitrary such that Ruv. From Definition fil.1(2a), we have $R^*[u][v]$, so that $\mathfrak{M}^*, [v] \models \varphi$; by inductive hypothesis $\mathfrak{M}, v \models \varphi$, and since v was arbitrary, $\mathfrak{M}, u \models \Box\varphi$. □

Corollary fil.3. *Let Γ be closed under subformulas. Then:*

1. *If $\mathfrak{M}^*$ is a filtration of $\mathfrak{M}$ through Γ then for any $\varphi \in \Gamma$: $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{M}^* \models \varphi$.*
2. *If $\mathcal{C}$ is a class of models and $\Gamma(\mathcal{C})$ is the class of Γ-filtrations of models in $\mathcal{C}$, then any formula $\varphi \in \Gamma$ is valid in $\mathcal{C}$ if and only if it is valid in $\Gamma(\mathcal{C})$.*

Photo Credits

Bibliography