### Filtrations

Rather than define “the” filtration of $\mathcal{M}$ through $\Gamma$, we define when a model $\mathcal{M}^*$ counts as a filtration of $\mathcal{M}$. All filtrations have the same set of worlds $W^*$ and the same valuation $V^*$. But different filtrations may have different accessibility relations $R^*$. To count as a filtration, $R^*$ has to satisfy a number of conditions, however. These conditions are exactly what we’ll require to prove the main result, namely that $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}^*, [w] \models \varphi$, provided $\varphi \in \Gamma$.

**Definition fil.1.** Let $\Gamma$ be closed under subformulas and $\mathcal{M} = \langle W, R, V \rangle$. A filtration of $\mathcal{M} \ through \ \Gamma$ is any model $\mathcal{M}^* = \langle W^*, R^*, V^* \rangle$, where:

1. $W^* = \{[w] : w \in W\}$;
2. For any $u, v \in W$:
   a) If $Ruv$ then $R^*[u][v]$;
   b) If $R^*[u][v]$ then for any $\square \varphi \in \Gamma$, if $\mathcal{M}, u \models \varphi$ then $\mathcal{M}, v \models \varphi$;
   c) If $R^*[u][v]$ then for any $\diamond \varphi \in \Gamma$, if $\mathcal{M}, v \models \varphi$ then $\mathcal{M}, u \models \varphi$.

It’s worthwhile thinking about what $V^*(p)$ is: the set consisting of the equivalence classes $[w]$ of all worlds $w$ where $p$ is true in $\mathcal{M}$. On the one hand, if $w \in V(p)$, then $[w] \in V^*(p)$ by that definition. However, it is not necessarily the case that if $[w] \in V^*(p)$, then $w \in V(p)$. If $[w] \in V^*(p)$ we are only guaranteed that $[w] = [u]$ for some $u \in V(p)$. Of course, $[w] = [u]$ means that $w \equiv u$. So, when $[w] \in V^*(p)$ we can (only) conclude that $w \equiv u$ for some $u \in V(p)$.

**Theorem fil.2.** If $\mathcal{M}^*$ is a filtration of $\mathcal{M} \ through \ \Gamma$, then for every $\varphi \in \Gamma$ and $w \in W$, we have $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}^*, [w] \models \varphi$.

**Proof.** By induction on $\varphi$, using the fact that $\Gamma$ is closed under subformulas. Since $\varphi \in \Gamma$ and $\Gamma$ is closed under sub-formulas, all sub-formulas of $\varphi$ are also in $\Gamma$. Hence in each inductive step, the induction hypothesis applies to the sub-formulas of $\varphi$.

1. $\varphi \equiv \bot$: Neither $\mathcal{M}, w \models \varphi$ nor $\mathcal{M}^*, [w] \models \varphi$.
2. $\varphi \equiv \top$: Both $\mathcal{M}, w \models \varphi$ and $\mathcal{M}^*, [w] \models \varphi$.
3. $\varphi \equiv p$: The left-to-right direction is immediate, as $\mathcal{M}, w \models \varphi$ only if $w \in V(p)$, which implies $[w] \in V^*(p)$, i.e., $\mathcal{M}^*, [w] \models \varphi$. Conversely, suppose $\mathcal{M}^*, [w] \models \varphi$, i.e., $[w] \in V^*(p)$. Then for some $v \in V(p)$, $w \equiv v$. Of course then also $\mathcal{M}, v \models p$. Since $w \equiv v$, $w$ and $v$ make the same formulas from $\Gamma$ true. Since by assumption $p \in \Gamma$ and $\mathcal{M}, v \models p$, $\mathcal{M}, w \models \varphi$. 

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Let $\text{Corollary fil.3.}$
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and validity in a class of models.

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Problem fil.1. Suppose $M, w \models \varphi$; to show that $M^*, [w] \models \varphi$, let $v$ be such that $R^*(w)[v]$. From Definition fil.1(2b), we have that $M, v \models \varphi$, and by inductive hypothesis $M^*, [v] \models \varphi$. Since $v$ was arbitrary, $M^*, [w] \models \varphi$ follows.

Conversely, suppose $M^*, [w] \models \varphi$ and let $v$ be arbitrary such that $Rwv$. From Definition fil.1(2a), we have $R^*(w)[v]$, so that $M^*, [v] \models \varphi$; by inductive hypothesis $M, v \models \varphi$, and since $v$ was arbitrary, $M, u \models \varphi$.

10. $\varphi \equiv \Diamond \psi$: Suppose $M, w \models \varphi$. Then for some $v \in W$, $Rwv$ and $M, v \models \psi$. By inductive hypothesis $M^*, [v] \models \psi$, and by Definition fil.1(2a), we have $R^*(w)[v]$. Thus, $M^*, [w] \models \varphi$.

Now suppose $M^*, [w] \models \varphi$. Then for some $[v] \in W^*$ with $R^*[w][v]$, $M^*, [v] \models \psi$. By inductive hypothesis $M, v \models \psi$. By Definition fil.1(2c), we have that $M, w \models \varphi$.

$\square$

Problem fil.1. Complete the proof of Theorem fil.2
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What holds for truth at worlds in a model also holds for truth in a model and validity in a class of models.

Corollary fil.3. Let $\Gamma$ be closed under subformulas. Then:

1. If $M^*$ is a filtration of $M$ through $\Gamma$ then for any $\varphi \in \Gamma$: $M \models \varphi$ if and only if $M^* \models \varphi$.

2. If $C$ is a class of models and $\Gamma(C)$ is the class of $\Gamma$-filtrations of models in $C$, then any formula $\varphi \in \Gamma$ is valid in $C$ if and only if it is valid in $\Gamma(C)$.

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Bibliography