1 Examples of Filtrations

We have not yet shown that there are any filtrations. But indeed, for any model \( M \), there are many filtrations of \( M \) through \( \Gamma \). We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as \( \Diamond \) stipulates directly what \( W^* \) and \( V^* \) should be). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible.

Definition fil.1. Where \( \Gamma \) is closed under subformulas, the finest filtration \( M^* \) of a model \( M \) is defined by putting:

\[
R^*[u][v] \quad \text{if and only if} \quad \exists u' \in [u] \exists v' \in [v] : Ru'v'.
\]

Proposition fil.2. The finest filtration \( M^* \) is indeed a filtration.

Proof. We need to check that \( R^* \), so defined, satisfies \( \Box \). We check the three conditions in turn.

If \( Ruv \), then since \( u \in [u] \) and \( v \in [v] \), also \( R^*[u][v] \), so \( \Box \) is satisfied.

For \( \Box \), suppose \( \Box \varphi \in \Gamma \), \( R^*[u][v] \), and \( M, u \vdash \Box \varphi \). By definition of \( R^* \), there are \( u' \equiv u \) and \( v' \equiv v \) such that \( Ru'v' \). Since \( u \) and \( u' \) agree on \( \Gamma \), also \( M, u' \vdash \Box \varphi \), so that \( M, v' \vdash \varphi \). By closure of \( \Gamma \) under sub-formulas, \( v \) and \( v' \) agree on \( \varphi \), so \( M, v \vdash \varphi \), as desired.

To verify \( \Diamond \), suppose \( \Diamond \varphi \in \Gamma \), \( R^*[u][v] \), and \( M, v \vdash \varphi \). By definition of \( R^* \), there are \( u' \equiv u \) and \( v' \equiv v \) such that \( Ru'v' \). Since \( v \) and \( v' \) agree on \( \Gamma \), and \( \Gamma \) is closed under sub-formulas, also \( M, v' \vdash \varphi \), so that \( M, u' \vdash \Diamond \varphi \). Since \( u \) and \( u' \) also agree on \( \Gamma \), \( M, u \vdash \Diamond \varphi \). □

Problem fil.1. Complete the proof of Proposition fil.2.

Definition fil.3. Where \( \Gamma \) is closed under subformulas, the coarsest filtration \( M^* \) of a model \( M \) is defined by putting \( R^*[u][v] \) if and only if both of the following conditions are met:

1. If \( \Box \varphi \in \Gamma \) and \( M, u \vdash \Box \varphi \) then \( M, v \vdash \varphi \);
2. If \( \Diamond \varphi \in \Gamma \) and \( M, v \vdash \varphi \) then \( M, u \vdash \Diamond \varphi \).

Proposition fil.4. The coarsest filtration \( M^* \) is indeed a filtration.

Proof. Given the definition of \( R^* \), the only condition that is left to verify is the implication from \( Ruv \) to \( R^*[u][v] \). So assume \( Ruv \). Suppose \( \Box \varphi \in \Gamma \) and \( M, u \vdash \Box \varphi \); then obviously \( M, v \vdash \varphi \), and (1) is satisfied. Suppose \( \Diamond \varphi \in \Gamma \) and \( M, v \vdash \varphi \). Then \( M, u \vdash \Diamond \varphi \) since \( Ruv \), and (2) is satisfied. □

Example fil.5. Let \( W = Z^+ \), \( Rnm \) iff \( m = n + 1 \), and \( V(p) = \{2n : n \in N\} \). The model \( M = (W, R, V) \) is depicted in Figure 1. The worlds are 1, 2, etc.; each world can access exactly one other world—its successor—and \( p \) is true at all and only the even numbers.

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Figure 1: An infinite model and its filtrations.

Now let \( \Gamma \) be the set of sub-formulas of \( \Box p \rightarrow p \), i.e., \( \{ p, \Box p, \Box p \rightarrow p \} \). \( p \) is true at all and only the even numbers, \( \Box p \) is true at all and only the odd numbers, so \( \Box p \rightarrow p \) is true at all and only the even numbers. In other words, every odd number makes \( \Box p \) true and \( p \) and \( \Box p \rightarrow p \) false; every even number makes \( p \) and \( \Box p \rightarrow p \) true, but \( \Box p \) false. So \( W^* = \{ [1], [2] \} \), where \( [1] = \{ 1, 3, 5, \ldots \} \) and \( [2] = \{ 2, 4, 6, \ldots \} \). Since \( 2 \in V(p) \), \( [2] \in V^*(p) \); since \( 1 \notin V(p) \), \( [1] \notin V^*(p) \). So \( V^*(p) = \{ [2] \} \).

Any filtration based on \( W^* \) must have an accessibility relation that includes \( \langle [1], [2] \rangle \), \( \langle [2], [1] \rangle \): since \( R_{12} \), we must have \( R^*[1][2] \) by ????, and since \( R_{23} \) we must have \( R^*[2][3] \), and \( [3] = [1] \). It cannot include \( \langle [1], [1] \rangle \): if it did, we’d have \( R^*[1][1] \), \( M, 1 \models \Box p \) but \( M, 1 \not\models p \), contradicting ???. Nothing requires or rules out that \( R^*[2][2] \). So, there are two possible filtrations of \( M \), corresponding to the two accessibility relations

\[
\{ \langle [1], [2] \rangle, \langle [2], [1] \rangle \} \text{ and } \{ \langle [1], [2] \rangle, \langle [2], [1] \rangle, \langle [2], [2] \rangle \}.
\]

In either case, \( p \) and \( \Box p \rightarrow p \) are false and \( \Box p \) is true at \([1] \); \( p \) and \( \Box p \rightarrow p \) are true and \( \Box p \) is false at \([2] \).

**Problem fil.2.** Consider the following model \( \mathcal{M} = (W, R, V) \) where \( W = \{ 0\sigma : \sigma \in \mathbb{B}^* \} \), the set of sequences of 0s and 1s starting with 0, with \( R\sigma\sigma' \) iff \( \sigma' = \sigma 0 \) or \( \sigma' = \sigma 1 \), and \( V(p) = \{ \sigma 0 : \sigma \in \mathbb{B}^* \} \) and \( V(q) = \{ \sigma 1 : \sigma \in \mathbb{B}^* \setminus \{ 1 \} \} \). Here’s a picture:
We have $\mathfrak{M}, w \not\models \Box (p \lor q) \rightarrow (\Box p \lor \Box q)$ for every $w$.

Let $\Gamma$ be the set of sub-formulas of $\Box (p \lor q) \rightarrow (\Box p \lor \Box q)$. What are $W^*$ and $V^*$? What is the accessibility relation of the finest filtration of $\mathfrak{M}$? Of the coarsest?

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Bibliography