fil.1 Examples of Filtrations

We have not yet shown that there are any filtrations. But indeed, for any model $\mathcal{M}$, there are many filtrations of $\mathcal{M}$ through $\Gamma$. We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as $\equiv$ stipulates directly what $W^*$ and $V^*$ should be). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible.

Definition fil.1. Where $\Gamma$ is closed under subformulas, the finest filtration $\mathcal{M}^*$ of a model $\mathcal{M}$ is defined by putting:

$$R^*[u][v] \quad \text{if and only if} \quad \exists u' \in [u] \exists v' \in [v] : Ruv' .$$

Proposition fil.2. The finest filtration $\mathcal{M}^*$ is indeed a filtration.

Proof. We need to check that $R^*$, so defined, satisfies $\forall \varphi \forall \varphi' \varphi \equiv \varphi'$. We check the three conditions in turn.

If $Ruv$ then since $u \in [u]$ and $v \in [v]$, also $R^*[u][v]$, so $\forall \varphi \forall \varphi' \varphi \lor \varphi'$. For $\forall \varphi$, suppose $\square \varphi \in \Gamma$, so $R^*[u][v]$, and $\mathcal{M}, u \models \square \varphi$. By definition of $R^*$, there are $u' \equiv u$ and $v' \equiv v$ such that $Ruv'$. Since $u$ and $u'$ agree on $\Gamma$, also $\mathcal{M}, u' \models \square \varphi'$, so that $\mathcal{M}, v' \models \varphi$. By closure of $\Gamma$ under subformulas, $v$ and $v'$ agree on $\varphi$, so $\mathcal{M}, v \models \varphi$, as desired.

To verify $\forall \varphi$, suppose $\Diamond \varphi \in \Gamma$, so $R^*[u][v]$, and $\mathcal{M}, v \models \varphi$. By definition of $R^*$, there are $u' \equiv u$ and $v' \equiv v$ such that $Ruv'$. Since $v$ and $v'$ agree on $\Gamma$, and $\Gamma$ is closed under subformulas, also $\mathcal{M}, v' \models \varphi$, so that $\mathcal{M}, u' \models \Diamond \varphi$. Since $u$ and $u'$ also agree on $\Gamma$, $\mathcal{M}, u \models \Diamond \varphi$. $\Box$

Problem fil.1. Complete the proof of Proposition fil.2.

Definition fil.3. Where $\Gamma$ is closed under subformulas, the coarsest filtration $\mathcal{M}^*$ of a model $\mathcal{M}$ is defined by putting $R^*[u][v]$ if and only if both of the following conditions are met:

1. If $\square \varphi \in \Gamma$ and $\mathcal{M}, u \models \square \varphi$ then $\mathcal{M}, v \models \varphi$;
2. If $\Diamond \varphi \in \Gamma$ and $\mathcal{M}, v \models \varphi$ then $\mathcal{M}, u \models \Diamond \varphi$.

Proposition fil.4. The coarsest filtration $\mathcal{M}^*$ is indeed a filtration.

Proof. Given the definition of $R^*$, the only condition that is left to verify is the implication from $Ruv$ to $R^*[u][v]$. So assume $Ruv$. Suppose $\square \varphi \in \Gamma$ and $\mathcal{M}, u \models \square \varphi$; then obviously $\mathcal{M}, v \models \varphi$, and (1) is satisfied. Suppose $\Diamond \varphi \in \Gamma$ and $\mathcal{M}, v \models \varphi$. Then $\mathcal{M}, u \models \Diamond \varphi$ since $Ruv$; and (2) is satisfied. $\Box$

Example fil.5. Let $W = \mathbb{Z}^+$, $Rnm$ iff $m = n + 1$, and $V(p) = \{2n : n \in \mathbb{N}\}$. The model $\mathcal{M} = \langle W, R, V \rangle$ is depicted in Figure 1. The worlds are $1, 2, \ldots$; each world can access exactly one other world—its successor—and $p$ is true at all and only the even numbers.
Now let $\Gamma$ be the set of sub-formulas of $\square p \rightarrow p$, i.e., \{p, \square p, \square p \rightarrow p\}. $p$ is true at all and only the even numbers, $\square p$ is true at all and only the odd numbers, so $\square p \rightarrow p$ is true at all and only the even numbers. In other words, every odd number makes $\square p$ true and $p$ and $\square p \rightarrow p$ false; every even number makes $p$ and $\square p \rightarrow p$ true, but $\square p$ false. So $W^* = \{[1], [2]\}$, where $[1] = \{1, 3, 5, \ldots\}$ and $[2] = \{2, 4, 6, \ldots\}$. Since $2 \in V(p)$, $[2] \in V^*(p)$; since $1 \notin V(p)$, $[1] \notin V^*(p)$. So $V^*(p) = \{[2]\}$.

Any filtration based on $W^*$ must have an accessibility relation that includes $\langle [1], [2]\rangle, \langle [2], [1]\rangle$: since $R_{12}$, we must have $R^*[1][2]$ by ????, and since $R_{23}$ we must have $R^*[2][3]$, and $[3] = [1]$. It cannot include $\langle [1], [1]\rangle$: if it did, we’d have $R^*[1][1]$, $M, 1 \models \square p$ but $M, 1 \not\models p$, contradicting ???. Nothing requires or rules out that $R^*[2][2]$. So, there are two possible filtrations of $M$, corresponding to the two accessibility relations

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\{\langle [1], [2]\rangle, \langle [2], [1]\rangle\} \text{ and } \{\langle [1], [2]\rangle, \langle [2], [1]\rangle, \langle [2], [2]\rangle\}.
\]

In either case, $p$ and $\square p \rightarrow p$ are false and $\square p$ is true at $[1]$; $p$ and $\square p \rightarrow p$ are true and $\square p$ is false at $[2]$.

**Problem fil.2.** Consider the following model $M = \langle W, R, V \rangle$ where $W = \{0\sigma : \sigma \in B^*\}$, the set of sequences of 0s and 1s starting with 0, with $R\sigma\sigma'$ iff $\sigma' = \sigma 0$ or $\sigma' = \sigma 1$, and $V(p) = \{\sigma 0 : \sigma \in B^*\}$ and $V(q) = \{\sigma 1 : \sigma \in B^* \setminus \{1\}\}$. Here’s a picture:
We have $\mathfrak{M}, w \not\vDash \Box(p \lor q) \rightarrow (\Box p \lor \Box q)$ for every $w$.

Let $\Gamma$ be the set of sub-formulas of $\Box(p \lor q) \rightarrow (\Box p \lor \Box q)$. What are $W^*$ and $V^*$? What is the accessibility relation of the finest filtration of $\mathfrak{M}$? Of the coarsest?

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Bibliography