Examples of Filtrations

We have not yet shown that there are any filtrations. But indeed, for any model $M$, there are many filtrations of $M$ through $I$. We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as ?? stipulates directly what $W^*$ and $V^*$ should be). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible.

Definition fil.1. Where $I$ is closed under subformulas, the finest filtration $M^*$ of a model $M$ is defined by putting:

$$R^*[u][v]$$ if and only if $\exists u' \in [u] \exists v' \in [v]: Ru'v'.

Proposition fil.2. The finest filtration $M^*$ is indeed a filtration.

Proof. We need to check that $R^*$, so defined, satisfies ????. We check the three conditions in turn.

If $Ruv$ then since $u \in [u]$ and $v \in [v]$, also $R^*[u][v]$, so ?? is satisfied.

For ??, suppose $\square \varphi \in I$, $R^*[u][v]$, and $M, u \vdash \square \varphi$. By definition of $R^*$, there are $u' \equiv u$ and $v' \equiv v$ such that $Ru'v'$. Since $u$ and $u'$ agree on $I$, also $M, u' \vdash \square \varphi$, so that $M, v' \vdash \varphi$. By closure of $I$ under sub-formulas, $v$ and $v'$ agree on $\varphi$, so $M, v \vdash \varphi$, as desired.

To verify ??, suppose $\Diamond \varphi \in I$, $R^*[u][v]$, and $M, v \vdash \varphi$. By definition of $R^*$, there are $u' \equiv u$ and $v' \equiv v$ such that $Ru'v'$. Since $v$ and $v'$ agree on $I$, and $I$ is closed under sub-formulas, also $M, v' \vdash \varphi$, so that $M, u' \vdash \Diamond \varphi$. Since $u$ and $u'$ also agree on $I$, $M, u \vdash \Diamond \varphi$.

Problem fil.1. Complete the proof of Proposition fil.2.

Definition fil.3. Where $I$ is closed under subformulas, the coarsest filtration $M^*$ of a model $M$ is defined by putting $R^*[u][v]$ if and only if both of the following conditions are met:

1. If $\square \varphi \in I$ and $M, u \vdash \square \varphi$ then $M, v \vdash \varphi$;
2. If $\Diamond \varphi \in I$ and $M, v \vdash \varphi$ then $M, u \vdash \Diamond \varphi$.

Proposition fil.4. The coarsest filtration $M^*$ is indeed a filtration.

Proof. Given the definition of $R^*$, the only condition that is left to verify is the implication from $Ruv$ to $R^*[u][v]$. So assume $Ruv$. Suppose $\square \varphi \in I$ and $M, u \vdash \square \varphi$; then obviously $M, v \vdash \varphi$, and (1) is satisfied. Suppose $\Diamond \varphi \in I$ and $M, v \vdash \varphi$. Then $M, u \vdash \Diamond \varphi$ since $Ruv$, and (2) is satisfied.

Example fil.5. Let $W = \mathbb{Z}^+$, $R_{nm}$ if $m = n + 1$, and $V(p) = \{2n : n \in \mathbb{N}\}$. The model $M = \langle W, R, V \rangle$ is depicted in Figure 1. The worlds are 1, 2, etc.; each world can access exactly one other world—its successor—and $p$ is true at all and only the even numbers.

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Now let $\Gamma$ be the set of sub-formulas of $\Box p \rightarrow p$, i.e., \{p, \Box p, \Box p \rightarrow p\}. p$ is true at all and only the even numbers, $\Box p$ is true at all and only the odd numbers, so $\Box p \rightarrow p$ is true at all and only the even numbers. In other words, every odd number makes $\Box p$ true and $p$ and $\Box p \rightarrow p$ false; every even number makes $p$ and $\Box p \rightarrow p$ true, but $\Box p$ false. So $W^* = \{[1], [2]\}$, where $[1] = \{1, 3, 5, \ldots \}$ and $[2] = \{2, 4, 6, \ldots \}$. Since $2 \in V(p)$, $[2] \in V^*(p)$; since $1 \notin V(p)$, $[1] \notin V^*(p)$. So $V^*(p) = \{[2]\}$.

Any filtration based on $W^*$ must have an accessibility relation that includes $\langle [1], [2] \rangle$, $\langle [2], [1] \rangle$: since $R_{12}$, we must have $R^*[1][2]$ by ????, and since $R_{23}$ we must have $R^*[2][3]$, and $[3] = [1]$. It cannot include $\langle [1], [1] \rangle$: if it did, we’d have $R^*[1][1]$, $\mathfrak{M}, 1 \Vdash \Box p$ but $\mathfrak{M}, 1 \nvdash p$, contradicting ???. Nothing requires or rules out that $R^*[2][2]$. So, there are two possible filtrations of $\mathfrak{M}$, corresponding to the two accessibility relations

$$\{\langle [1], [2] \rangle, \langle [2], [1] \rangle\} \text{ and } \{\langle [1], [2] \rangle, \langle [2], [1] \rangle, \langle [2], [2] \rangle\}.$$

In either case, $p$ and $\Box p \rightarrow p$ are false and $\Box p$ is true at $[1]$; $p$ and $\Box p \rightarrow p$ are true and $\Box p$ is false at $[2]$.

**Problem fil.2.** Consider the following model $\mathfrak{M} = (W, R, V)$ where $W = \{0\sigma : \sigma \in B^*\}$, the set of sequences of 0s and 1s starting with 0, with $R\sigma\sigma'$ iff $\sigma' = \sigma0$ or $\sigma' = \sigma1$, and $V(p) = \{\sigma0 : \sigma \in B^*\}$ and $V(q) = \{\sigma1 : \sigma \in B^* \setminus \{1\}\}$. Here’s a picture:
We have $\mathcal{M}, w \not\models □(p \lor q) \rightarrow (□p \lor □q)$ for every $w$.

Let $Γ$ be the set of sub-formulas of $□(p \lor q) \rightarrow (□p \lor □q)$. What are $W^*$ and $V^*$? What is the accessibility relation of the finest filtration of $\mathcal{M}$? Of the coarsest?

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Bibliography