fil.1 Examples of Filtrations

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We have not yet shown that there are any filtrations. But indeed, for any model $\mathfrak{M}$, there are many filtrations of $\mathfrak{M}$ through Γ. We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as **??** stipulates directly what W^* and V^* should be like). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible.

Definition fil.1. Where Γ is closed under subformulas, the *finest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting:

$$R^*[u][v] \quad \text{if and only if} \quad \exists u' \in [u]\, \exists v' \in [v] : Ru'v'.$$

Proposition fil.2. *The finest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Proof. We need to check that R^*, so defined, satisfies **????**. We check the three conditions in turn.

If Ruv then by reflexivity of $\equiv$, also $R^*[u][v]$, so **??** is satisfied.

For **??**, suppose $\Box\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, u \models \Box\varphi$. By definition of R^*, there are $u' \equiv u$ an $v' \equiv v$ such that $Ru'v'$. Since u and u' agree on Γ, also $\mathfrak{M}, u' \models \Box\varphi$, so that $\mathfrak{M}, v' \models \varphi$. By closure of Γ, v and v' agree on φ, so $\mathfrak{M}, v \models \varphi$, as desired.

To verify **??**, suppose $\Diamond\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, v \models \varphi$. Arguing similarly to the previous case, $\mathfrak{M}, u \models \Diamond\varphi$. □

Definition fil.3. Where Γ is closed under subformulas, the *coarsest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting $R^*[u][v]$ if and only if *both* of the following conditions are met:

1. If $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$ then $\mathfrak{M}, v \models \varphi$;
2. If $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$.

Proposition fil.4. *The coarsest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Proof. Given the definition of R^*, the only condition that is left to verify is the implication from Ruv to $R^*[u][v]$. Assuming Ruv, suppose $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \models \Box\varphi$; then obviously $\mathfrak{M}, v \models \varphi$. Similarly if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, u \models \Diamond\varphi$. □

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Bibliography