fil.1 Examples of Filtrations

mod:fil:exf: sec

We have not yet shown that there are any filtrations. But indeed, for any model $\mathfrak{M}$, there are many filtrations of $\mathfrak{M}$ through Γ. We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as **??** stipulates directly what W^* and V^* should be). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible.

Definition fil.1. Where Γ is closed under subformulas, the *finest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting:

$$R^*[u][v] \quad \text{if and only if} \quad \exists u' \in [u]\, \exists v' \in [v] : Ru'v'.$$

mod:fil:exf: prop:finest

Proposition fil.2. *The finest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Proof. We need to check that R^*, so defined, satisfies **????**. We check the three conditions in turn.

If Ruv then since $u \in [u]$ and $v \in [v]$, also $R^*[u][v]$, so **??** is satisfied.

For **??**, suppose $\Box\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, u \Vdash \Box\varphi$. By definition of R^*, there are $u' \equiv u$ and $v' \equiv v$ such that $Ru'v'$. Since u and u' agree on Γ, also $\mathfrak{M}, u' \Vdash \Box\varphi$, so that $\mathfrak{M}, v' \Vdash \varphi$. By closure of Γ under sub-formulas, v and v' agree on φ, so $\mathfrak{M}, v \Vdash \varphi$, as desired.

To verify **??**, suppose $\Diamond\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, v \Vdash \varphi$. By definition of R^*, there are $u' \equiv u$ and $v' \equiv v$ such that $Ru'v'$. Since v and v' agree on Γ, and Γ is closed under sub-formulas, also $\mathfrak{M}, v' \Vdash \varphi$, so that $\mathfrak{M}, u' \Vdash \Diamond\varphi$. Since u and u' also agree on Γ, $\mathfrak{M}, u \Vdash \Diamond\varphi$. □

Problem fil.1. Complete the proof of Proposition fil.2.

Definition fil.3. Where Γ is closed under subformulas, the *coarsest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting $R^*[u][v]$ if and only if *both* of the following conditions are met:

mod:fil:exf: defn:coarsest-Box

1. If $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$ then $\mathfrak{M}, v \Vdash \varphi$;

mod:fil:exf: defn:coarsest-Diamond

2. If $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \varphi$ then $\mathfrak{M}, u \Vdash \Diamond\varphi$.

Proposition fil.4. *The coarsest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Proof. Given the definition of R^*, the only condition that is left to verify is the implication from Ruv to $R^*[u][v]$. So assume Ruv. Suppose $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$; then obviously $\mathfrak{M}, v \Vdash \varphi$, and (1) is satisfied. Suppose $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \varphi$. Then $\mathfrak{M}, u \Vdash \Diamond\varphi$ since Ruv, and (2) is satisfied. □

Example fil.5. Let $W = \mathbb{Z}^+$, Rnm iff $m = n + 1$, and $V(p) = \{2n : n \in \mathbb{N}\}$. The model $\mathfrak{M} = \langle W, R, V \rangle$ is depicted in Figure 1. The worlds are 1, 2, etc.; each world can access exactly one other world—its successor, and p is true at all and only the even numbers.

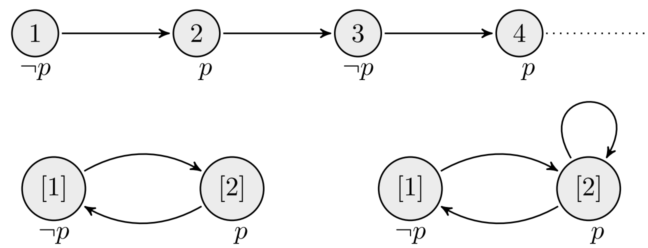


Figure 1: An infinite model and its filtrations.

mod:fil:exf: fig:ex-filtration

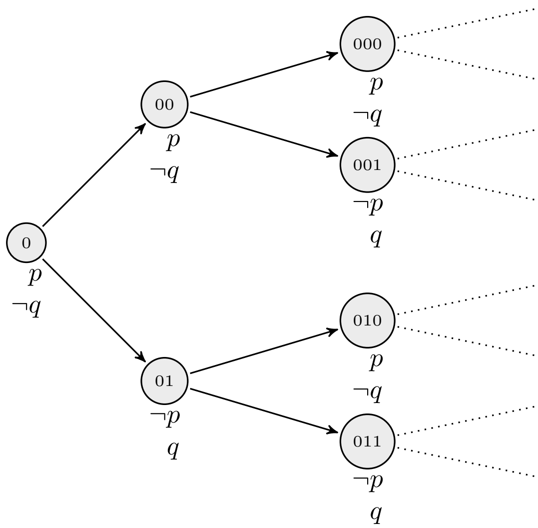
Now let Γ be the set of sub-formulas of $\Box p \to p$, i.e., $\{p, \Box p, \Box p \to p\}$. p is true at all and only the even numbers, $\Box p$ is true at all and only the odd numbers, so $\Box p \to p$ is true at all and only the even numbers. In other words, every odd number makes $\Box p$ true and p and $\Box p \to p$ false; every even number makes p and $\Box p \to p$ true, but $\Box p$ false. So $W^* = \{[1], [2]\}$, where $[1] = \{1, 3, 5, \dots\}$ and $[2] = \{2, 4, 6, \dots\}$. Since $2 \in V(p)$, $[2] \in V^*(p)$; since $1 \notin V(p)$, $[1] \notin V^*(p)$. So $V^*(p) = \{[2]\}$.

Any filtration based on W^* must have an accessibility relation that includes $\langle [1], [2] \rangle, \langle [2], [1] \rangle$: since $R12$, we must have $R^*[1][2]$ by **????**, and since $R23$ we must have $R^*[2][3]$, and $[3] = [1]$. It cannot include $\langle [1], [1] \rangle$: if it did, we'd have $R^*[1][1]$, $\mathfrak{M}, 1 \Vdash \Box p$ but $\mathfrak{M}, 1 \Vdash p$, contradicting **??**. Nothing requires or rules out that $R^*[2][2]$. So, there are two possible filtrations of $\mathfrak{M}$, corresponding to the two accessibility relations

$$\{\langle [1], [2] \rangle, \langle [2], [1] \rangle\} \text{ and } \{\langle [1], [2] \rangle, \langle [2], [1] \rangle, \langle [2], [2] \rangle\}.$$

In either case, p and $\Box p \to p$ are false and $\Box p$ is true at $[1]$; p and $\Box p \to p$ are true and $\Box p$ is false at $[2]$.

Problem fil.2. Consider the following model $\mathfrak{M} = \langle W, R, V \rangle$ where $W = \mathbb{B}^* \setminus \{1\sigma : \sigma \in \mathbb{B}^*\} \setminus \{\Lambda\}$, the set of sequences of 0s and 1s starting with 0, with $R\sigma\sigma'$ iff $\sigma' = \sigma 0$ or $\sigma' = \sigma 1$, and $V(p) = \{\sigma 0 : \sigma \in \mathbb{B}^*\}$ and $V(q) = \{\sigma 1 : \sigma \in \mathbb{B}^* \setminus \{1\}\}$. Here's a picture:



We have $\mathfrak{M}, w \nVdash \Box(p \vee q) \to (\Box p \vee \Box q)$ for every w.

Let Γ be the set of sub-formulas of $\Box(p \vee q) \to (\Box p \vee \Box q)$. What are W^* an V^*? What is the accessibility relation of the finest filtration of $\mathfrak{M}$? Of the coarsest?

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Bibliography