fil.1 Filtrations of Euclidean Models

nml:fil:euc:sec

The approach of **??** does not work in the case of models that are euclidean or serial and euclidean. Consider the model at the top of Figure 1, which is both euclidean and serial. Let $\Gamma = \{p, \Box p\}$. When taking a filtration through Γ, then $[w_1] = [w_3]$ since w_1 and w_3 are the only worlds that agree on Γ. Any filtration will also have the arrow inherited from $\mathfrak{M}$, as depicted in Figure 2. That model isn't euclidean. Moreover, we cannot add arrows to that model in order to make it euclidean. We would have to add double arrows between $[w_2]$ and $[w_4]$, and then also between w_2 and w_5. But $\Box p$ is supposed to be true at w_2, while p is false at w_5.

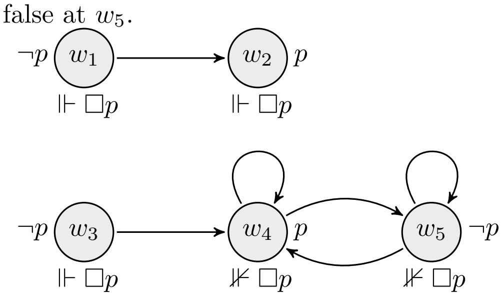

Figure 1: A serial and euclidean model.

nml:fil:euc:fig:ser-eucl

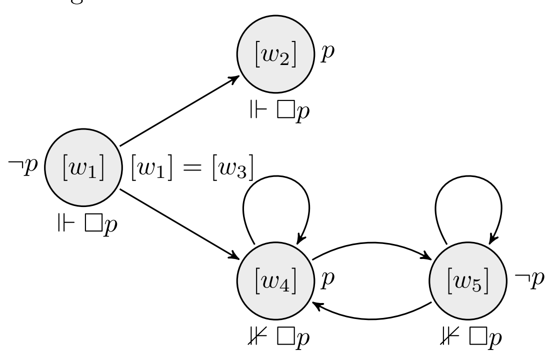

Figure 2: The filtration of the model in Figure 1.

nml:fil:euc:fig:ser-eucl2

In particular, to obtain a euclidean filtration it is not enough to consider filtrations through arbitrary Γ's closed under sub-formulas. Instead we need to consider sets Γ that are *modally closed* (see **??**). Such sets of sentences are infinite, and therefore do not immediately yield a finite model property or the decidability of the corresponding system.

nml:fil:euc:thm:modal-closed-filt

Theorem fil.1. *Let Γ be modally closed, $\mathfrak{M} = \langle W, R, V\rangle$, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ be a coarsest filtration of $\mathfrak{M}$.*

1. *If $\mathfrak{M}$ is symmetric, so is $\mathfrak{M}^*$.*
2. *If $\mathfrak{M}$ is transitive, so is $\mathfrak{M}^*$.*
3. *If $\mathfrak{M}$ is euclidean, so is $\mathfrak{M}^*$.*

Proof. 1. If $\mathfrak{M}^*$ is a coarsest filtration, then by definition $R^*[u][v]$ holds if and only if $C_1(u, v)$. For transitivity, suppose $C_1(u, v)$ and $C_1(v, w)$; we have to show $C_1(u, w)$. Suppose $\mathfrak{M}, u \Vdash \Box\varphi$; then $\mathfrak{M}, u \Vdash \Box\Box\varphi$ since 4 is valid in all transitive models; since $\Box\Box\varphi \in \Gamma$ by closure, also by $C_1(u, v)$, $\mathfrak{M}, v \Vdash \Box\varphi$ and by $C_1(v, w)$, also $\mathfrak{M}, w \Vdash \varphi$. Suppose $\mathfrak{M}, w \Vdash \varphi$; then $\mathfrak{M}, v \Vdash \Diamond\varphi$ by $C_1(v, w)$, since $\Diamond\varphi \in \Gamma$ by modal closure. By $C_1(u, v)$, we get $\mathfrak{M}, u \Vdash \Diamond\Diamond\varphi$ since $\Diamond\Diamond\varphi \in \Gamma$ by modal closure. Since $4_\Diamond$ is valid in all transitive models, $\mathfrak{M}, u \Vdash \Diamond\varphi$.

2. Exercise. Use the fact that both 5 and $5_\Diamond$ are valid in all euclidean models.

3. Exercise. Use the fact that B and $\mathrm{B}_\Diamond$ are valid in all symmetric models. □

Problem fil.1. Complete the proof of Theorem fil.1.

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Bibliography