**fil.1** K and S5 have the Finite Model Property

**Definition fil.1.** A system $\Sigma$ of modal logic is said to have the *finite model property* if whenever a formula $\varphi$ is true at a world in a model of $\Sigma$ then $\varphi$ is true at a world in a *finite* model of $\Sigma$.

**Proposition fil.2.** K has the *finite model property*.

*Proof.* K is the set of valid formulas, i.e., any model is a model of K. By ??, if $\mathfrak{M}, w \models \varphi$, then $\mathfrak{M}^\ast, w \models \varphi$ for any filtration of $\mathfrak{M}$ through the set $\Gamma$ of sub-formulas of $\varphi$. Any formula only has finitely many sub-formulas, so $\Gamma$ is finite. By ??, $|W^\ast| \leq 2^n$, where $n$ is the number of formulas in $\Gamma$. And since K imposes no restriction on models, $\mathfrak{M}^\ast$ is a K-model.

To show that a logic L has the finite model property via filtrations it is essential that the filtration of an L-model is itself a L-model. Often this requires a fair bit of work, and not any filtration yields a L-model. However, for universal models, this still holds.

**Proposition fil.3.** Let $\mathcal{U}$ be the class of universal models (see ??) and $\mathcal{U}_{\text{Fin}}$ the class of all finite universal models. Then any formula $\varphi$ is valid in $\mathcal{U}$ if and only if it is valid in $\mathcal{U}_{\text{Fin}}$.

*Proof.* Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to left direction, suppose that $\varphi$ is false at some world $w$ in a universal model $\mathfrak{M}$. Let $\Gamma$ contain $\varphi$ as well as all of its subformulas; clearly $\Gamma$ is finite. Take a filtration $\mathfrak{M}^\ast$ of $\mathfrak{M}$; then $\mathfrak{M}^\ast$ is finite by ??, and by ??, $\varphi$ is false at $[w]$ in $\mathfrak{M}^\ast$. It remains to observe that $\mathfrak{M}^\ast$ is also universal: given $u$ and $v$, by hypothesis $Ruv$ and by ????, also $R^\ast[u][v]$.

**Corollary fil.4.** S5 has the finite model property.

*Proof.* By ??, if $\varphi$ is true at a world in some reflexive and euclidean model then it is true at a world in a universal model. By Proposition fil.3, it is true at a world in a finite universal model (namely the filtration of the model through the set of sub-formulas of $\varphi$). Every universal model is also reflexive and euclidean; so $\varphi$ is true at a world in a finite reflexive euclidean model.

**Problem fil.1.** Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

**Problem fil.2.** Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.