fil.1 K and S5 have the Finite Model Property

nml:fil:fmp: sec

Definition fil.1. A system Σ of modal logic is said to have the *finite model property* if whenever a formula φ is true at a world in a model of Σ then φ is true at a world in a *finite* model of Σ.

nml:fil:fmp: prop:K-fmp **Proposition fil.2.** **K** *has the finite model property.*

Proof. **K** is the set of valid formulas, i.e., any model is a model of **K**. By **??**, if $\mathfrak{M}, w \Vdash \varphi$, then $\mathfrak{M}^*, w \Vdash \varphi$ for any filtration of $\mathfrak{M}$ through the set Γ of sub-formulas of φ. Any formula only has finitely many sub-formulas, so Γ is finite. By **??**, $|W^*| \leq 2^n$, where n is the number of formulas in Γ. And since **K** imposes no restriction on models, $\mathfrak{M}^*$ is a **K**-model. □

To show that a logic **L** has the finite model property via filtrations it is essential that the filtration of an **L**-model is itself a **L**-model. Often this requires a fair bit of work, and not any filtration yields a **L**-model. However, for universal models, this still holds.

nml:fil:fmp: prop:univ-fin **Proposition fil.3.** *Let $\mathcal{U}$ be the class of universal models (see* **??***) and $\mathcal{U}_{\mathrm{Fin}}$ the class of all finite universal models. Then any formula φ is valid in $\mathcal{U}$ if and only if it is valid in $\mathcal{U}_{\mathrm{Fin}}$.*

Proof. Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to left direction, suppose that φ is false at some world w in a universal model $\mathfrak{M}$. Let Γ contain φ as well as all of its subformulas; clearly Γ is finite. Take a filtration $\mathfrak{M}^*$ of $\mathfrak{M}$; then $\mathfrak{M}^*$ is finite by **??**, and by **??**, φ is false at $[w]$ in $\mathfrak{M}^*$. It remains to observe that $\mathfrak{M}^*$ is also universal: given u and v, by hypothesis Ruv and by **????**, also $R^*[u][v]$. □

nml:fil:fmp: cor:S5fmp **Corollary fil.4.** **S5** *has the finite model property.*

Proof. By **??**, if φ is true at a world in some reflexive and euclidean model then it is true at a world in a universal model. By Proposition fil.3, it is true at a world in a finite universal model (namely the filtration of the model through the set of sub-formulas of φ). Every universal model is also reflexive and euclidean; so φ is true at a world in a finite reflexive euclidean model. □

Problem fil.1. Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

Problem fil.2. Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

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Bibliography