fil.1  K and S5 have the Finite Model Property

Definition fil.1. A system $\Sigma$ of modal logic is said to have the finite model property if whenever a formula $\varphi$ is true at a world in a model of $\Sigma$ then $\varphi$ is true at a world in a finite model of $\Sigma$.

Proposition fil.2. $K$ has the finite model property.

Proof. $K$ is the set of valid formulas, i.e., any model is a model of $K$. By ??, if $\mathfrak{M}, w \models \varphi$, then $\mathfrak{M}^*, w \models \varphi$ for any filtration of $\mathfrak{M}$ through the set $\Gamma$ of sub-formulas of $\varphi$. Any formula only has finitely many sub-formulas, so $\Gamma$ is finite. By ??, $|W^*| \leq 2^n$, where $n$ is the number of formulas in $\Gamma$. And since $K$ imposes no restriction on models, $\mathfrak{M}^*$ is a $K$-model.

To show that a logic $L$ has the finite model property via filtrations it is essential that the filtration of an $L$-model is itself a $L$-model. Often this requires a fair bit of work, and not any filtration yields a $L$-model. However, for universal models, this still holds.

Proposition fil.3. Let $\mathcal{U}$ be the class of universal models (see ??) and $\mathcal{U}_{\text{Fin}}$ the class of all finite universal models. Then any formula $\varphi$ is valid in $\mathcal{U}$ if and only if it is valid in $\mathcal{U}_{\text{Fin}}$.

Proof. Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to left direction, suppose that $\varphi$ is false at some world $w$ in a universal model $\mathfrak{M}$. Let $\Gamma$ contain $\varphi$ as well as all of its subformulas; clearly $\Gamma$ is finite. Take a filtration $\mathfrak{M}^*$ of $\mathfrak{M}$; then $\mathfrak{M}^*$ is finite by ??, and by ??, $\varphi$ is false at $[w]$ in $\mathfrak{M}^*$. It remains to observe that $\mathfrak{M}^*$ is also universal: given $u$ and $v$, by hypothesis $Ruv$ and by ???, also $R^*[u][v]$.

Corollary fil.4. S5 has the finite model property.

Proof. By ??, if $\varphi$ is true at a world in some reflexive and euclidean model then it is true at a world in a universal model. By Proposition fil.3, it is true at a world in a finite universal model (namely the filtration of the model through the set of sub-formulas of $\varphi$). Every universal model is also reflexive and euclidean; so $\varphi$ is true at a world in a finite reflexive euclidean model.

Problem fil.1. Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

Problem fil.2. Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.
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Bibliography