

## fil.1 K and S5 have the Finite Model Property

mod:fil:fmp:  
sec

**Definition fil.1.** A system  $\Sigma$  of modal logic is said to have the *finite model property* if whenever a formula  $\varphi$  is true at a world in a model of  $\Sigma$  then  $\varphi$  is true at a world in a *finite* model of  $\Sigma$ .

mod:fil:fmp:  
prop:K-fmp

**Proposition fil.2.** **K** has the finite model property.

*Proof.* **K** is the set of valid formulas, i.e., any model is a model of **K**. By ??, if  $\mathfrak{M}\varphi[w]$ , then  $\mathfrak{M}^*\varphi[w]$  for any filtration of  $\mathfrak{M}$  through the set  $\Gamma$  of sub-formulas of  $\varphi$ . Any formula only has finitely many sub-formulas, so  $\Gamma$  is finite. By ??,  $|W^*| \leq 2^n$ , where  $n$  is the number of formulas in  $\Gamma$ . And since **K** imposes no restriction on models,  $\mathfrak{M}^*$  is a **K**-model.  $\square$

To show that a logic **L** has the finite model property via filtrations it is essential that the filtration of an **L**-model is itself a **L**-model. Often this requires a fair bit of work, and not any filtration yields a **L**-model. However, for universal models, this still holds.

mod:fil:fmp:  
prop:univ-fin

**Proposition fil.3.** Let  $\mathcal{U}$  be the class of universal models (see ??) and  $\mathcal{U}_{\text{Fin}}$  the class of all finite universal models. Then any formula  $\varphi$  is valid in  $\mathcal{U}$  if and only if it is valid in  $\mathcal{U}_{\text{Fin}}$ .

*Proof.* Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to left direction, suppose that  $\varphi$  is false at some world  $w$  in a universal model  $\mathfrak{M}$ . Let  $\Gamma$  contain  $\varphi$  as well as all of its subformulas; clearly  $\Gamma$  is finite. Take a filtration  $\mathfrak{M}^*$  of  $\mathfrak{M}$ ; then  $\mathfrak{M}^*$  is finite by ??, and by ??,  $\varphi$  is false at  $[w]$  in  $\mathfrak{M}^*$ . It remains to observe that  $\mathfrak{M}^*$  is also universal: given  $u$  and  $v$ , by hypothesis  $Ruv$  and by Definition ?????, also  $R^*[u][v]$ .  $\square$

mod:fil:fmp:  
cor:S5fmp

**Corollary fil.4.** **S5** has the finite model property.

*Proof.* By ??, if  $\varphi$  is true at a world in some reflexive and euclidean model then it is true at a world in a universal model. By Proposition fil.3, it is true at a world in a finite universal model (namely the filtration of the model through the set of sub-formulas of  $\varphi$ ). Every universal model is also reflexive and euclidean; so  $\varphi$  is true at a world in a finite reflexive euclidean model.  $\square$

**Problem fil.1.** Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

**Problem fil.2.** Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

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**Bibliography**