com.1 The Truth Lemma

mod:com:tru:sec

mod:com:tru:prop:truthlemma **Proposition com.1** (Truth Lemma)**.** *For every formula φ, $\mathfrak{M}^\Sigma, w \models \varphi$ if and only if $\varphi \in w$.*

Proof. By induction on φ. *Basis*: if φ is a propositional variable, say p, then:

$$\mathfrak{M}^\Sigma, w \models p \Leftrightarrow w \in V^\Sigma(p) \Leftrightarrow p \in w.$$

If φ is $\bot$ then both $\mathfrak{M}^\Sigma, w \not\models \bot$ and $\bot \notin w$ (by consistency of w). The cases for $\neg\varphi$ and $\varphi \to \psi$ follow from the inductive hypothesis and **??**, parts **??** and **??**. Here is the case for $\Box\varphi$; in one direction:

$$\begin{aligned}
\mathfrak{M}^\Sigma, w \models \Box\varphi \quad &\Rightarrow \quad \forall w' \in W^\Sigma(R^\Sigma ww' \Rightarrow \mathfrak{M}^\Sigma, w' \models \varphi), && \text{def. } \models;\\
&\Rightarrow \quad \forall w' \in W^\Sigma(\{\psi : \Box\psi \in w\} \subseteq w' \Rightarrow \mathfrak{M}^\Sigma, w' \models \varphi), && \text{def. } R^\Sigma;\\
&\Rightarrow \quad \forall w' \in W^\Sigma(\{\psi : \Box\psi \in w\} \subseteq w' \Rightarrow \varphi \in w'), && \text{ind. hyp.};\\
&\Rightarrow \quad \Box\varphi \in w, && \textbf{??}.
\end{aligned}$$

Conversely, assume $\Box\varphi \in w$, and let w' be an arbitrary world in W^Σ such that $R^\Sigma ww'$. By definition of R^Σ, we have $\{\psi : \Box\psi \in w\} \subseteq w'$, which immediately gives $\varphi \in w'$. By induction hypothesis, $\mathfrak{M}^\Sigma, w' \models \varphi$, and since w' was arbitrary, $\mathfrak{M}^\Sigma, w \models \Box\varphi$. □

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Bibliography