

## com.1 The Truth Lemma

mod:com:tru:  
sec

The canonical model  $\mathfrak{M}^\Sigma$  is defined in such a way that  $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$  iff  $\varphi \in \Delta$ . For propositional variables, the definition of  $V^\Sigma$  yields this directly. We have to verify that the equivalence holds for all **formulas**, however. We do this by induction. The inductive step involves proving the equivalence for **formulas** involving propositional operators (where we have to use **??**) and the modal operators (where we invoke the results of **??**).

mod:com:tru:  
prop:truthlemma

**Proposition com.1** (Truth Lemma). *For every **formula**  $\varphi$ ,  $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$  if and only if  $\varphi \in \Delta$ .*

*Proof.* By induction on  $\varphi$ .

1.  $\varphi \equiv \perp$ :  $\mathfrak{M}^\Sigma, \Delta \not\Vdash \perp$  by **??**, and  $\perp \notin \Delta$  by **????**.
2.  $\varphi \equiv \top$ :  $\mathfrak{M}^\Sigma, \Delta \Vdash \top$  by **??**, and  $\top \in \Delta$  by **????**.
3.  $\varphi \equiv p$ :  $\mathfrak{M}^\Sigma, \Delta \Vdash p$  iff  $\Delta \in V^\Sigma(p)$  by **??**. Also,  $\Delta \in V^\Sigma(p)$  iff  $p \in \Delta$  by definition of  $V^\Sigma$ .
4.  $\varphi \equiv \neg\psi$ :  $\mathfrak{M}^\Sigma, \Delta \Vdash \neg\psi$  iff  $\mathfrak{M}^\Sigma, \Delta \not\Vdash \psi$  (**??**) iff  $\psi \notin \Delta$  (by inductive hypothesis) iff  $\neg\psi \in \Delta$  (by **????**).
5.  $\varphi \equiv \psi \wedge \chi$ :  $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \wedge \chi$  iff  $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$  and  $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$  (by **??**) iff  $\psi \in \Delta$  and  $\chi \in \Delta$  (by inductive hypothesis) iff  $\psi \wedge \chi \in \Delta$  (by **????**).
6.  $\varphi \equiv \psi \vee \chi$ :  $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \vee \chi$  iff  $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$  or  $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$  (by **??**) iff  $\psi \in \Delta$  or  $\chi \in \Delta$  (by inductive hypothesis) iff  $\psi \vee \chi \in \Delta$  (by **????**).
7.  $\varphi \equiv \psi \rightarrow \chi$ :  $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \rightarrow \chi$  iff  $\mathfrak{M}^\Sigma, \Delta \not\Vdash \psi$  or  $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$  (by **??**) iff  $\psi \notin \Delta$  or  $\chi \in \Delta$  (by inductive hypothesis) iff  $\psi \rightarrow \chi \in \Delta$  (by **????**).
8.  $\varphi \equiv \psi \leftrightarrow \chi$ :  $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \leftrightarrow \chi$  iff either  $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$  and  $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$  or  $\mathfrak{M}^\Sigma, \Delta \not\Vdash \psi$  and  $\mathfrak{M}^\Sigma, \Delta \not\Vdash \chi$  (by **??**) iff either  $\psi \in \Delta$  and  $\chi \in \Delta$  or  $\psi \notin \Delta$  and  $\chi \notin \Delta$  (by inductive hypothesis) iff  $\psi \leftrightarrow \chi \in \Delta$  (by **????**).
9.  $\varphi \equiv \Box\psi$ : First suppose that  $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$ . By **??**, for every  $\Delta'$  such that  $R^\Sigma \Delta \Delta'$ ,  $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$ . By inductive hypothesis, for every  $\Delta'$  such that  $R^\Sigma \Delta \Delta'$ ,  $\psi \in \Delta'$ . By definition of  $R^\Sigma$ , for every  $\Delta'$  such that  $\Box^{-1}\Delta \subseteq \Delta'$ ,  $\psi \in \Delta'$ . By **??**,  $\Box\psi \in \Delta$ .  
Now assume  $\Box\psi \in \Delta$ . Let  $\Delta' \in W^\Sigma$  be such that  $R^\Sigma \Delta \Delta'$ , i.e.,  $\Box^{-1}\Delta \subseteq \Delta'$ . Since  $\Box\psi \in \Delta$ ,  $\psi \in \Box^{-1}\Delta$ . Consequently,  $\psi \in \Delta'$ . By inductive hypothesis,  $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$ . Since  $\Delta'$  is arbitrary with  $R^\Sigma \Delta \Delta'$ , for all  $\Delta' \in W^\Sigma$  such that  $R^\Sigma \Delta \Delta'$ ,  $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$ . By **??**,  $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$ .
10.  $\varphi \equiv \Diamond\psi$ : First suppose that  $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\psi$ . By **??**, for some  $\Delta'$  such that  $R^\Sigma \Delta \Delta'$ ,  $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$ . By inductive hypothesis, for some  $\Delta'$  such that  $R^\Sigma \Delta \Delta'$ ,  $\psi \in \Delta'$ . By definition of  $R^\Sigma$ , for some  $\Delta'$  such that  $\Box^{-1}\Delta \subseteq \Delta'$ ,

$\psi \in \Delta'$ . By ??, for some  $\Delta'$  such that  $\diamond\Delta' \subseteq \Delta$ ,  $\psi \in \Delta'$ . Since  $\psi \in \Delta'$ ,  $\diamond\psi \in \diamond\Delta'$ , so  $\diamond\psi \in \Delta$ .

Now assume  $\diamond\psi \in \Delta$ . By ??, there is a complete  $\Sigma$ -consistent  $\Delta' \in W^\Sigma$  be such that  $\diamond\Delta' \subseteq \Delta$  and  $\psi \in \Delta'$ . By ??, there is a  $\Delta' \in W^\Sigma$  such that  $\square^{-1}\Delta \subseteq \Delta'$ , and  $\psi \in \Delta'$ . By definition of  $R^\Sigma$ ,  $R^\Sigma\Delta\Delta'$ , so there is a  $\Delta' \in W^\Sigma$  such that  $R^\Sigma\Delta\Delta'$  and  $\psi \in \Delta'$ . By ??,  $\mathfrak{M}^\Sigma, \Delta \Vdash \diamond\psi$ .

□

**Problem com.1.** Complete the proof of [Proposition com.1](#).

## Photo Credits

## Bibliography