com.1 The Truth Lemma

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The canonical model $\mathfrak{M}^\Sigma$ is defined in such a way that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. For propositional variables, the definition of V^Σ yields this directly. We have to verify that the equivalence holds for all formulas, however. We do this by induction. The inductive step involves proving the equivalence for formulas involving propositional operators (where we have to use **??**) and the modal operators (where we invoke the results of **??**).

nml:com:tru: prop:truthlemma

Proposition com.1 (Truth Lemma). *For every formula φ, $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ if and only if $\varphi \in \Delta$.*

Proof. By induction on φ.

1. $\varphi \equiv \bot$: $\mathfrak{M}^\Sigma, \Delta \nVdash \bot$ by **??**, and $\bot \notin \Delta$ by **????**.

2. $\varphi \equiv \top$: $\mathfrak{M}^\Sigma, \Delta \Vdash \top$ by **??**, and $\top \in \Delta$ by **????**.

3. $\varphi \equiv p$: $\mathfrak{M}^\Sigma, \Delta \Vdash p$ iff $\Delta \in V^\Sigma(p)$ by **??**. Also, $\Delta \in V^\Sigma(p)$ iff $p \in \Delta$ by definition of V^Σ.

4. $\varphi \equiv \neg\psi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \neg\psi$ iff $\mathfrak{M}^\Sigma, \Delta \nVdash \psi$ (**??**) iff $\psi \notin \Delta$ (by inductive hypothesis) iff $\neg\psi \in \Delta$ (by **????**).

5. $\varphi \equiv \psi \land \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \land \chi$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \in \Delta$ and $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \land \chi \in \Delta$ (by **????**).

6. $\varphi \equiv \psi \lor \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \lor \chi$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ or $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \in \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \lor \chi \in \Delta$ (by **????**).

7. $\varphi \equiv \psi \to \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \to \chi$ iff $\mathfrak{M}^\Sigma, \Delta \nVdash \psi$ or $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \notin \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \to \chi \in \Delta$ (by **????**).

8. $\varphi \equiv \psi \leftrightarrow \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \leftrightarrow \chi$ iff either $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ or $\mathfrak{M}^\Sigma, \Delta \nVdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \nVdash \chi$ (by **??**) iff either $\psi \in \Delta$ and $\chi \in \Delta$ or $\psi \notin \Delta$ and $\chi \notin \Delta$ (by inductive hypothesis) iff $\psi \leftrightarrow \chi \in \Delta$ (by **????**).

9. $\varphi \equiv \Box\psi$: First suppose that $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$. By **??**, for every Δ' such that $R^\Sigma\Delta\Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By inductive hypothesis, for every Δ' such that $R^\Sigma\Delta\Delta'$, $\psi \in \Delta'$. By definition of R^Σ, for every Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$, $\psi \in \Delta'$. By **??**, $\Box\psi \in \Delta$.

   Now assume $\Box\psi \in \Delta$. Let $\Delta' \in W^\Sigma$ be such that $R^\Sigma\Delta\Delta'$, i.e., $\Box^{-1}\Delta \subseteq \Delta'$. Since $\Box\psi \in \Delta$, $\psi \in \Box^{-1}\Delta$. Consequently, $\psi \in \Delta'$. By inductive hypothesis, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. Since Δ' is arbitrary with $R^\Sigma\Delta\Delta'$, for all $\Delta' \in W^\Sigma$ such that $R^\Sigma\Delta\Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By **??**, $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$.

10. $\varphi \equiv \Diamond\psi$: First suppose that $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\psi$. By **??**, for some Δ' such that $R^\Sigma\Delta\Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By inductive hypothesis, for some Δ' such that $R^\Sigma\Delta\Delta'$, $\psi \in \Delta'$. By definition of R^Σ, for some Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$,

$\psi \in \Delta'$. By **??**, for some Δ' such that $\Diamond\Delta' \subseteq \Delta$, $\psi \in \Delta'$. Since $\psi \in \Delta'$, $\Diamond\psi \in \Diamond\Delta'$, so $\Diamond\psi \in \Delta$.

Now assume $\Diamond\psi \in \Delta$. By **??**, there is a complete Σ-consistent $\Delta' \in W^{\Sigma}$ such that $\Diamond\Delta' \subseteq \Delta$ and $\psi \in \Delta'$. By **??**, there is a $\Delta' \in W^{\Sigma}$ such that $\Box^{-1}\Delta \subseteq \Delta'$, and $\psi \in \Delta'$. By definition of R^{Σ}, $R^{\Sigma}\Delta\Delta'$, so there is a $\Delta' \in W^{\Sigma}$ such that $R^{\Sigma}\Delta\Delta'$ and $\psi \in \Delta'$. By **??**, $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Diamond\psi$. □

Problem com.1. Complete the proof of Proposition com.1.

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Bibliography