com.1 Derivability and Complete Consistent Sets

mod:com:pcc:sec

mod:com:pcc:cor:provability-characterization **Corollary com.1.** *$\Gamma \vdash_\Sigma \varphi$ if and only if $\varphi \in \Delta$ for each complete Σ-consistent set Δ extending Γ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system Σ.)*

Proof. Suppose $\Gamma \vdash_\Sigma \varphi$, and let Δ be any complete Σ-consistent set extending Γ. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_\Sigma \varphi$ (by monotony) and $\Delta \vdash_\Sigma \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \nvdash_\Sigma \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent, and by Lindenbaum's Lemma there is a complete consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$. By consistency, $\varphi \notin \Delta$. □

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Bibliography