Modalities and Complete Consistent Sets

When we construct a model \( M^\Sigma \) whose set of worlds is given by the complete \( \Sigma \)-consistent sets \( \Delta \) in some normal modal logic \( \Sigma \), we will also need to define an accessibility relation \( R^\Sigma \) between such “worlds.” We want it to be the case that the accessibility relation (and the assignment \( V^\Sigma \)) are defined in such a way that \( M^\Sigma, \Delta \models \varphi \) iff \( \varphi \in \Delta \). How should we do this?

Once the accessibility relation is defined, the definition of truth at a world ensures that \( M^\Sigma, \Delta \models \Box \varphi \) iff \( M^\Sigma, \Delta' \models \varphi \) for all \( \Delta' \) such that \( R^\Sigma \Delta \Delta' \). The proof that \( M^\Sigma, \Delta \models \varphi \) iff \( \varphi \in \Delta \) requires that this is true in particular for formulas starting with a modal operator, i.e., \( M^\Sigma, \Delta \models \Box \varphi \) iff \( \Box \varphi \in \Delta \). Combining this requirement with the definition of truth at a world for \( \Box \varphi \) yields:

\[
\Box \varphi \in \Delta \text{ iff } \varphi \in \Delta' \text{ for all } \Delta' \text{ with } R^\Sigma \Delta \Delta'
\]

Consider the left-to-right direction: it says that if \( \Box \varphi \in \Delta \), then \( \varphi \in \Delta' \) for any \( \varphi \) and any \( \Delta' \) with \( R^\Sigma \Delta \Delta' \). If we stipulate that \( R^\Sigma \Delta \Delta' \) iff \( \Box \varphi \in \Delta \) for all \( \Box \varphi \in \Delta \), then this holds. We can write the condition on the right of the “iff” more compactly as: \( \{ \varphi : \Box \varphi \in \Delta \} \subseteq \Delta' \).

So the question is: does this definition of \( R^\Sigma \) in fact guarantee that \( \Box \varphi \in \Delta \) iff \( M^\Sigma, \Delta \models \Box \varphi \)? Does it also guarantee that \( \Diamond \varphi \in \Delta \) iff \( M^\Sigma, \Delta \models \Diamond \varphi \)? The next few results will establish this.

**Definition com.1.** If \( \Gamma \) is a set of formulas, let

\[
\Box \Gamma = \{ \Box \psi : \psi \in \Gamma \}
\]

\[
\Diamond \Gamma = \{ \Diamond \psi : \psi \in \Gamma \}
\]

and

\[
\Box^{-1} \Gamma = \{ \psi : \Box \psi \in \Gamma \}
\]

\[
\Diamond^{-1} \Gamma = \{ \psi : \Diamond \psi \in \Gamma \}
\]

In other words, \( \Box \Gamma \) is \( \Gamma \) with \( \Box \) in front of every formula in \( \Gamma \); \( \Box^{-1} \Gamma \) is all the \( \Box \)ed formulas of \( \Gamma \) with the initial \( \Box \)'s removed. This definition is not terribly important on its own, but will simplify the notation considerably.

Note that \( \Box \Box^{-1} \Gamma \subseteq \Gamma \):

\[
\Box \Box^{-1} \Gamma = \{ \Box \psi : \Box \psi \in \Gamma \}
\]

i.e., it’s just the set of all those formulas of \( \Gamma \) that start with \( \Box \).

**Lemma com.2.** If \( \Gamma \vdash \Sigma \varphi \) then \( \Box \Gamma \vdash \Sigma \Box \varphi \).

*Proof.** If \( \Gamma \vdash \Sigma \varphi \) then there are \( \psi_1, \ldots, \psi_k \in \Gamma \) such that \( \Sigma \vdash \psi_1 \rightarrow (\psi_2 \rightarrow \cdots (\psi_n \rightarrow \varphi) \cdots) \). Since \( \Sigma \) is normal, by rule RK, \( \Sigma \vdash \Box \psi_1 \rightarrow (\Box \psi_2 \rightarrow \cdots (\Box \psi_n \rightarrow \Box \varphi) \cdots) \), where obviously \( \Box \psi_1, \ldots, \Box \psi_k \in \Box \Gamma \). Hence, by definition, \( \Box \Gamma \vdash \Sigma \Box \varphi \). \( \square \)
Lemma com.3. If $\Box^{-1} \Gamma \vdash \Sigma \varphi$ then $\Gamma \vdash \Theta \varphi$.

Proof. Suppose $\Box^{-1} \Gamma \vdash \Sigma \varphi$; then by Lemma com.2, $\Theta \Box^{-1} \Gamma \vdash \Theta \varphi$. But since $\Theta \Box^{-1} \Gamma \subseteq \Gamma$, also $\Gamma \vdash \Theta \varphi$ by monotonicity.

Proposition com.4. If $\Gamma$ is complete $\Sigma$-consistent, then $\Box \varphi \in \Gamma$ if and only if for every complete $\Sigma$-consistent $\Delta$ such that $\Box^{-1} \Gamma \subseteq \Delta$, it holds that $\varphi \in \Delta$.

Proof. Suppose $\Gamma$ is complete $\Sigma$-consistent. The “only if” direction is easy: Suppose $\Box \varphi \in \Gamma$ and that $\Box^{-1} \Gamma \subseteq \Delta$. Since $\Box \varphi \in \Gamma$, $\varphi \in \Box^{-1} \Gamma \subseteq \Delta$, so $\varphi \in \Delta$.

For the “if” direction, we prove the contrapositive: Suppose $\Box \varphi \notin \Gamma$. Since $\Gamma$ is complete $\Sigma$-consistent, it is deductively closed, and hence $\Gamma \not\vdash \Box \varphi$. By Lemma com.3, $\Box^{-1} \Gamma \vdash \Sigma \varphi$. By Lemma com.4, $\Box^{-1} \Gamma \cup \{\neg \varphi\}$ is $\Sigma$-consistent. By Lindenbaum’s Lemma, there is a complete $\Sigma$-consistent set $\Delta$ such that $\Box^{-1} \Gamma \cup \{\neg \varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$.

Lemma com.5. Suppose $\Gamma$ and $\Delta$ are complete $\Sigma$-consistent. Then $\Box^{-1} \Gamma \subseteq \Delta$ if and only if $\Diamond \Delta \subseteq \Gamma$.

Proof. “Only if” direction: Assume $\Box^{-1} \Gamma \subseteq \Delta$ and suppose $\Diamond \varphi \in \Diamond \Delta$ (i.e., $\varphi \in \Delta$). In order to show $\Diamond \varphi \in \Gamma$, it suffices to show $\Box \neg \varphi \notin \Gamma$, for then by maximality, $\neg \Box \neg \varphi \in \Gamma$. Now, if $\Box \neg \varphi \in \Gamma$ then by hypothesis $\neg \varphi \in \Delta$, against the consistency of $\Delta$ (since $\varphi \in \Delta$). Hence $\Box \neg \varphi \notin \Gamma$, as required.

“If” direction: Assume $\Diamond \Delta \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box \varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg \varphi \in \Delta$ and so by hypothesis $\Box \neg \varphi \in \Gamma$. But in a normal modal logic $\Box \neg \varphi$ is equivalent to $\neg \Box \varphi$, and if the latter is in $\Gamma$, by consistency $\Box \varphi \notin \Gamma$, as required.

Proposition com.6. If $\Gamma$ is complete $\Sigma$-consistent, then $\Diamond \varphi \in \Gamma$ if and only if for some complete $\Sigma$-consistent $\Delta$ such that $\Diamond \Delta \subseteq \Gamma$, it holds that $\varphi \in \Delta$.

Proof. Suppose $\Gamma$ is complete $\Sigma$-consistent. $\Diamond \varphi \in \Gamma$ iff $\neg \Box \neg \varphi \in \Gamma$ by dual and closure. $\neg \Box \neg \varphi \in \Gamma$ iff $\Box \neg \varphi \notin \Gamma$ by Lemma com.2 since $\Gamma$ is complete $\Sigma$-consistent. By Proposition com.4, $\Box \neg \varphi \notin \Gamma$ iff, for some complete $\Sigma$-consistent $\Delta$ with $\Box^{-1} \Gamma \subseteq \Delta$, $\neg \varphi \notin \Delta$. Now consider any such $\Delta$. By Lemma com.5, $\Box^{-1} \Gamma \subseteq \Delta$ iff $\Diamond \Delta \subseteq \Gamma$. Also, $\neg \varphi \notin \Delta$ iff $\varphi \in \Delta$ by Lemma com.4. So $\Diamond \varphi \in \Gamma$ iff, for some complete $\Sigma$-consistent $\Delta$ with $\Diamond \Delta \subseteq \Gamma$, $\varphi \in \Delta$.

Problem com.1. Show that if $\Gamma$ is complete $\Sigma$-consistent, then $\Diamond \varphi \in \Gamma$ if and only if there is a complete $\Sigma$-consistent $\Delta$ such that $\Box^{-1} \Gamma \subseteq \Delta$ and $\varphi \in \Delta$. Do this without using Lemma com.5.
Photo Credits

Bibliography