com.1 Modalities and Complete Consistent Sets

mod:com:mod: sec

When we construct a model whose set of worlds is given by the complete consistent sets in some normal modal logic Σ, we will also need to define an accessibility relation between such "worlds." The next few lemmas give us the tools to do so. As noted, Σ will be a normal modal logic throughout.

mod:com:mod: lem:Gamma-proves1

Lemma com.1. *If $\Gamma \vdash_{\Sigma} \varphi$ then $\{\Box\psi : \psi \in \Gamma\} \vdash_{\Sigma} \Box\varphi$.*

Proof. If $\Gamma \vdash_{\Sigma} \varphi$ then there are $\psi_1, \ldots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots(\psi_n \to \varphi)\cdots)$. Since Σ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \to (\Box\psi_2 \to \cdots(\Box\psi_n \to \Box\varphi)\cdots)$, where obviously $\Box\psi_1, \ldots, \Box\psi_k \in \{\Box\psi : \psi \in \Gamma\}$. Hence, by definition, $\{\Box\psi : \psi \in \Gamma\} \vdash_{\Sigma} \Box\varphi$. □

mod:com:mod: lem:Gamma-proves2

Lemma com.2. *If $\{\psi : \Box\psi \in \Gamma\} \vdash_{\Sigma} \varphi$ then $\Gamma \vdash_{\Sigma} \Box\varphi$.*

Proof. Let $\Delta = \{\psi : \Box\psi \in \Gamma\}$, so that $\Delta \vdash_{\Sigma} \varphi$; then by Lemma com.1, $\{\Box\psi : \psi \in \Delta\} \vdash \Box\varphi$. But obviously $\{\Box\psi : \psi \in \Delta\} \subseteq \Gamma$, so also $\Gamma \vdash_{\Sigma} \Box\varphi$ by Monotony. □

mod:com:mod: thm:box-phiGamma

Theorem com.3. *If Γ is complete Σ-consistent, then $\Box\varphi \in \Gamma$ if and only if for every complete Σ-consistent Δ such that $\{\psi : \Box\psi \in \Gamma\} \subseteq \Delta$, it holds that $\varphi \in \Delta$.*

Proof. The left-to-right half of the theorem is obvious. For the converse, suppose $\Box\varphi \notin \Gamma$. Since Γ is deductively closed, $\Gamma \nvdash_{\Sigma} \Box\varphi$, and by Lemma com.2 $\{\psi : \Box\psi \in \Gamma\} \nvdash_{\Sigma} \varphi$. By **????**, $\{\psi : \Box\psi \in \Gamma\} \cup \{\neg\varphi\}$ is Σ-consistent, so that by Lindenbaum's Lemma there is a complete Σ-consistent set Δ such that $\{\psi : \Box\psi \in \Gamma\} \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$, and the theorem is proved. □

mod:com:mod: lem:Gamma-proves3

Lemma com.4. *Suppose Γ and Δ are complete Σ-consistent. Then: $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$ if and only if $\{\Diamond\varphi : \varphi \in \Delta\} \subseteq \Gamma$.*

Proof. "Only if" direction: Assume $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$ and suppose $\varphi \in \Delta$. In order to show $\Diamond\varphi \in \Gamma$ it suffices to show $\Box\neg\varphi \notin \Gamma$ for then by maximality $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of Δ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

"If" direction: Assume $\{\Diamond\varphi : \varphi \in \Delta\} \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in Γ by consistency $\Box\varphi \notin \Gamma$, as required. □

Corollary com.5. *If Γ is complete Σ-consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some complete Σ-consistent Δ such that $\{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma$, it holds that $\varphi \in \Delta$.*

Proof. Suppose Γ is complete Σ-consistent, and argue as follows:

$$\begin{array}{lcll}
\Diamond\varphi \in \Gamma & \Leftrightarrow & \neg\Box\neg\varphi \in \Gamma, & \text{re-writing;} \\
& \Leftrightarrow & \Box\neg\varphi \notin \Gamma, & \Gamma \text{ is complete } \Sigma\text{-con} \\
& \Leftrightarrow & \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\psi : \Box\psi \in \Gamma\} \subseteq \Delta \wedge \neg\varphi \notin \Delta]\,, & \text{Theorem com.3;} \\
& \Leftrightarrow & \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma \wedge \neg\varphi \notin \Delta]\,, & \text{Lemma com.4;} \\
& \Leftrightarrow & \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma \wedge \varphi \in \Delta]\,, \Box & \Delta \text{ is complete.}
\end{array}$$

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Bibliography