## com.1 Modalities and Complete Consistent Sets

nml:com:mod:sec When we construct a model $\mathfrak{M}^{\Sigma}$ whose set of worlds is given by the complete $\Sigma$-consistent sets $\Delta$ in some normal modal logic $\Sigma$, we will also need to define an accessibility relation $R^{\Sigma}$ between such "worlds." We want it to be the case that the accessibility relation (and the assignment $V^{\Sigma}$) are defined in such a way that $\mathfrak{M}^{\Sigma}, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. How should we do this? explanation

Once the accessibility relation is defined, the definition of truth at a world ensures that $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Box\varphi$ iff $\mathfrak{M}^{\Sigma}, \Delta' \Vdash \varphi$ for all $\Delta'$ such that $R^{\Sigma}\Delta\Delta'$. The proof that $\mathfrak{M}^{\Sigma}, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$ requires that this is true in particular for formulas starting with a modal operator, i.e., $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Box\varphi$ iff $\Box\varphi \in \Delta$. Combining this requirement with the definition of truth at a world for $\Box\varphi$ yields:

$$\Box\varphi \in \Delta \text{ iff } \varphi \in \Delta' \text{ for all } \Delta' \text{ with } R^{\Sigma}\Delta\Delta'$$

Consider the left-to-right direction: it says that if $\Box\varphi \in \Delta$, then $\varphi \in \Delta'$ for any $\varphi$ and any $\Delta'$ with $R^{\Sigma}\Delta\Delta'$. If we stipulate that $R^{\Sigma}\Delta\Delta'$ iff $\varphi \in \Delta'$ for all $\Box\varphi \in \Delta$, then this holds. We can write the condition on the right of the "iff" more compactly as: $\{\varphi : \Box\varphi \in \Delta\} \subseteq \Delta'$.

So the question is: does this definition of $R^{\Sigma}$ in fact guarantee that $\Box\varphi \in \Delta$ iff $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Box\varphi$? Does it also guarantee that $\Diamond\varphi \in \Delta$ iff $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Diamond\varphi$? The next few results will establish this.

**Definition com.1.** If $\Gamma$ is a set of formulas, let

$$\Box\Gamma = \{\Box\psi : \psi \in \Gamma\}$$
$$\Diamond\Gamma = \{\Diamond\psi : \psi \in \Gamma\}$$

and

$$\Box^{-1}\Gamma = \{\psi : \Box\psi \in \Gamma\}$$
$$\Diamond^{-1}\Gamma = \{\psi : \Diamond\psi \in \Gamma\}$$

In other words, $\Box\Gamma$ is $\Gamma$ with $\Box$ in front of every formula in $\Gamma$; $\Box^{-1}\Gamma$ is all the $\Box$'ed formulas of $\Gamma$ with the initial $\Box$'s removed. This definition is not terribly important on its own, but will simplify the notation considerably.

Note that $\Box\Box^{-1}\Gamma \subseteq \Gamma$:

$$\Box\Box^{-1}\Gamma = \{\Box\psi : \Box\psi \in \Gamma\}$$

i.e., it's just the set of all those formulas of $\Gamma$ that start with $\Box$.

nml:com:mod:lem:box1 **Lemma com.2.** *If $\Gamma \vdash_{\Sigma} \varphi$ then $\Box\Gamma \vdash_{\Sigma} \Box\varphi$.*

*Proof.* If $\Gamma \vdash_{\Sigma} \varphi$ then there are $\psi_1, \dots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$. Since $\Sigma$ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \to (\Box\psi_2 \to \cdots (\Box\psi_n \to \Box\varphi) \cdots)$, where obviously $\Box\psi_1, \dots, \Box\psi_k \in \Box\Gamma$. Hence, by definition, $\Box\Gamma \vdash_{\Sigma} \Box\varphi$. □

**Lemma com.3.** *If $\Box^{-1}\Gamma \vdash_\Sigma \varphi$ then $\Gamma \vdash_\Sigma \Box\varphi$.* *nml:com:mod:lem:box2*

*Proof.* Suppose $\Box^{-1}\Gamma \vdash_\Sigma \varphi$; then by Lemma com.2, $\Box\Box^{-1}\Gamma \vdash \Box\varphi$. But since $\Box\Box^{-1}\Gamma \subseteq \Gamma$, also $\Gamma \vdash_\Sigma \Box\varphi$ by monotonicity. □

**Proposition com.4.** *If $\Gamma$ is complete $\Sigma$-consistent, then $\Box\varphi \in \Gamma$ if and only if for every complete $\Sigma$-consistent $\Delta$ such that $\Box^{-1}\Gamma \subseteq \Delta$, it holds that $\varphi \in \Delta$.* *nml:com:mod:prop:box*

*Proof.* Suppose $\Gamma$ is complete $\Sigma$-consistent. The "only if" direction is easy: Suppose $\Box\varphi \in \Gamma$ and that $\Box^{-1}\Gamma \subseteq \Delta$. Since $\Box\varphi \in \Gamma$, $\varphi \in \Box^{-1}\Gamma \subseteq \Delta$, so $\varphi \in \Delta$.

For the "if" direction, we prove the contrapositive: Suppose $\Box\varphi \notin \Gamma$. Since $\Gamma$ is complete $\Sigma$-consistent, it is deductively closed, and hence $\Gamma \nvdash_\Sigma \Box\varphi$. By Lemma com.3, $\Box^{-1}\Gamma \nvdash_\Sigma \varphi$. By **????**, $\Box^{-1}\Gamma \cup \{\neg\varphi\}$ is $\Sigma$-consistent. By Lindenbaum's Lemma, there is a complete $\Sigma$-consistent set $\Delta$ such that $\Box^{-1}\Gamma \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$. □

**Lemma com.5.** *Suppose $\Gamma$ and $\Delta$ are complete $\Sigma$-consistent. Then $\Box^{-1}\Gamma \subseteq \Delta$ if and only if $\Diamond\Delta \subseteq \Gamma$.* *nml:com:mod:lem:box-iff-diamond*

*Proof.* "Only if" direction: Assume $\Box^{-1}\Gamma \subseteq \Delta$ and suppose $\Diamond\varphi \in \Diamond\Delta$ (i.e., $\varphi \in \Delta$). In order to show $\Diamond\varphi \in \Gamma$, it suffices to show $\Box\neg\varphi \notin \Gamma$, for then by maximality, $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of $\Delta$ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

"If" direction: Assume $\Diamond\Delta \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in $\Gamma$, by consistency $\Box\varphi \notin \Gamma$, as required. □

**Proposition com.6.** *If $\Gamma$ is complete $\Sigma$-consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some complete $\Sigma$-consistent $\Delta$ such that $\Diamond\Delta \subseteq \Gamma$, it holds that $\varphi \in \Delta$.* *nml:com:mod:prop:diamond*

*Proof.* Suppose $\Gamma$ is complete $\Sigma$-consistent. $\Diamond\varphi \in \Gamma$ iff $\neg\Box\neg\varphi \in \Gamma$ by DUAL and closure. $\neg\Box\neg\varphi \in \Gamma$ iff $\Box\neg\varphi \notin \Gamma$ by **????** since $\Gamma$ is complete $\Sigma$-consistent. By Proposition com.4, $\Box\neg\varphi \notin \Gamma$ iff, for some complete $\Sigma$-consistent $\Delta$ with $\Box^{-1}\Gamma \subseteq \Delta$, $\neg\varphi \notin \Delta$. Now consider any such $\Delta$. By Lemma com.5, $\Box^{-1}\Gamma \subseteq \Delta$ iff $\Diamond\Delta \subseteq \Gamma$. Also, $\neg\varphi \notin \Delta$ iff $\varphi \in \Delta$ by **????**. So $\Diamond\varphi \in \Gamma$ iff, for some complete $\Sigma$-consistent $\Delta$ with $\Diamond\Delta \subseteq \Gamma$, $\varphi \in \Delta$. □

**Problem com.1.** Show that if $\Gamma$ is complete $\Sigma$-consistent, then $\Diamond\varphi \in \Gamma$ if and only if there is a complete $\Sigma$-consistent $\Delta$ such that $\Box^{-1}\Gamma \subseteq \Delta$ and $\varphi \in \Delta$. *Do this without using Lemma com.5.*

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## Bibliography