

com.1 Modalities and Complete Consistent Sets

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sec

When we construct a model \mathfrak{M}^Σ whose set of worlds is given by the complete Σ -consistent sets Δ in some normal modal logic Σ , we will also need to define an accessibility relation R^Σ between such “worlds.” We want it to be the case that the accessibility relation (and the assignment V^Σ) are defined in such a way that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. How should we do this?

explanation

Once the accessibility relation is defined, the definition of truth at a world ensures that $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$ iff $\mathfrak{M}^\Sigma, \Delta' \Vdash \varphi$ for all Δ' such that $R^\Sigma \Delta \Delta'$. The proof that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$ requires that this is true in particular for **formulas** starting with a modal operator, i.e., $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$ iff $\Box\varphi \in \Delta$. Combining this requirement with the definition of truth at a world for $\Box\varphi$ yields:

$$\Box\varphi \in \Delta \text{ iff } \varphi \in \Delta' \text{ for all } \Delta' \text{ with } R^\Sigma \Delta \Delta'$$

Consider the left-to-right direction: it says that if $\Box\varphi \in \Delta$, then $\varphi \in \Delta'$ for any φ and any Δ' with $R^\Sigma \Delta \Delta'$. If we stipulate that $R^\Sigma \Delta \Delta'$ iff $\varphi \in \Delta'$ for all $\Box\varphi \in \Delta$, then this holds. We can write the condition on the right of the “iff” more compactly as: $\{\varphi : \Box\varphi \in \Delta\} \subseteq \Delta'$.

So the question is: does this definition of R^Σ in fact guarantee that $\Box\varphi \in \Delta$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$? Does it also guarantee that $\Diamond\varphi \in \Delta$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\varphi$? The next few results will establish this.

Definition com.1. If Γ is a set of **formulas**, let

$$\begin{aligned} \Box\Gamma &= \{\Box\psi : \psi \in \Gamma\} \\ \Diamond\Gamma &= \{\Diamond\psi : \psi \in \Gamma\} \end{aligned}$$

and

$$\begin{aligned} \Box^{-1}\Gamma &= \{\psi : \Box\psi \in \Gamma\} \\ \Diamond^{-1}\Gamma &= \{\psi : \Diamond\psi \in \Gamma\} \end{aligned}$$

In other words, $\Box\Gamma$ is Γ with \Box in front of every **formula** in Γ ; $\Box^{-1}\Gamma$ is all the \Box 'ed **formulas** of Γ with the initial \Box 's removed. This definition is not terribly important on its own, but will simplify the notation considerably.

Note that $\Box\Box^{-1}\Gamma \subseteq \Gamma$:

$$\Box\Box^{-1}\Gamma = \{\Box\psi : \Box\psi \in \Gamma\}$$

i.e., it's just the set of all those **formulas** of Γ that start with \Box .

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lem:box1

Lemma com.2. *If $\Gamma \vdash_\Sigma \varphi$ then $\Box\Gamma \vdash_\Sigma \Box\varphi$.*

Proof. If $\Gamma \vdash_\Sigma \varphi$ then there are $\psi_1, \dots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \rightarrow (\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots)$. Since Σ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \rightarrow (\Box\psi_2 \rightarrow \dots (\Box\psi_n \rightarrow \Box\varphi) \dots)$, where obviously $\Box\psi_1, \dots, \Box\psi_k \in \Box\Gamma$. Hence, by definition, $\Box\Gamma \vdash_\Sigma \Box\varphi$. \square

Lemma com.3. *If $\Box^{-1}\Gamma \vdash_{\Sigma} \varphi$ then $\Gamma \vdash_{\Sigma} \Box\varphi$.*

*mod:com:mod:
lem:box2*

Proof. Suppose $\Box^{-1}\Gamma \vdash_{\Sigma} \varphi$; then by [Lemma com.2](#), $\Box\Box^{-1}\Gamma \vdash \Box\varphi$. But since $\Box\Box^{-1}\Gamma \subseteq \Gamma$, also $\Gamma \vdash_{\Sigma} \Box\varphi$ by Monotony. \square

Proposition com.4. *If Γ is complete Σ -consistent, then $\Box\varphi \in \Gamma$ if and only if for every complete Σ -consistent Δ such that $\Box^{-1}\Gamma \subseteq \Delta$, it holds that $\varphi \in \Delta$.*

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prop:box*

Proof. Suppose Γ is complete Σ -consistent. The “only if” direction is easy: Suppose $\Box\varphi \in \Gamma$ and that $\Box^{-1}\Gamma \subseteq \Delta$. Since $\Box\varphi \in \Gamma$, $\varphi \in \Box^{-1}\Gamma \subseteq \Delta$, so $\varphi \in \Delta$.

For the “if” direction, we prove the contrapositive: Suppose $\Box\varphi \notin \Gamma$. Since Γ is complete Σ -consistent, it is deductively closed, and hence $\Gamma \not\vdash_{\Sigma} \Box\varphi$. By [Lemma com.3](#), $\Box^{-1}\Gamma \not\vdash_{\Sigma} \varphi$. By [????](#), $\Box^{-1}\Gamma \cup \{\neg\varphi\}$ is Σ -consistent. By Lindenbaum’s Lemma, there is a complete Σ -consistent set Δ such that $\Box^{-1}\Gamma \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$. \square

Lemma com.5. *Suppose Γ and Δ are complete Σ -consistent. Then: $\Box^{-1}\Gamma \subseteq \Delta$ if and only if $\Diamond\Delta \subseteq \Gamma$.*

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lem:box-iff-diamond*

Proof. “Only if” direction: Assume $\Box^{-1}\Gamma \subseteq \Delta$ and suppose $\Diamond\varphi \in \Diamond\Delta$ (i.e., $\varphi \in \Delta$). In order to show $\Diamond\varphi \in \Gamma$ it suffices to show $\Box\neg\varphi \notin \Gamma$ for then by maximality $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of Δ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

“If” direction: Assume $\Diamond\Delta \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in Γ , by consistency $\Box\varphi \notin \Gamma$, as required. \square

Proposition com.6. *If Γ is complete Σ -consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some complete Σ -consistent Δ such that $\Diamond\Delta \subseteq \Gamma$, it holds that $\varphi \in \Delta$.*

*mod:com:mod:
prop:diamond*

Proof. Suppose Γ is complete Σ -consistent. $\Diamond\varphi \in \Gamma$ iff $\neg\Box\neg\varphi \in \Gamma$ by DUAL and closure. $\neg\Box\neg\varphi \in \Gamma$ iff $\Box\neg\varphi \notin \Gamma$ by [????](#) since Γ is complete Σ -consistent. By [Proposition com.4](#), $\Box\neg\varphi \notin \Gamma$ iff, for some complete Σ -consistent Δ with $\Box^{-1}\Gamma \subseteq \Delta$, $\neg\varphi \notin \Delta$. Now consider any such Δ . By [Lemma com.5](#), $\Box^{-1}\Gamma \subseteq \Delta$ iff $\Diamond\Delta \subseteq \Gamma$. Also, $\neg\varphi \notin \Delta$ iff $\varphi \in \Delta$ by [????](#). So $\Diamond\varphi \in \Gamma$ iff, for some complete Σ -consistent Δ with $\Diamond\Delta \subseteq \Gamma$, $\varphi \in \Delta$. \square

Problem com.1. Show that if Γ is complete Σ -consistent, then $\Diamond\varphi \in \Gamma$ if and only if there is a complete Σ -consistent Δ such that $\Box^{-1}\Gamma \subseteq \Delta$ and $\varphi \in \Delta$. Do this without using [Lemma com.5](#).

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Bibliography