com.1 Lindenbaum's Lemma

mod:com:lin: sec

Lindenbaum's Lemma establishes that every Σ-consistent set of formulas is contained in at least one *complete* Σ-consistent set. Our construction of the canonical model will show that for each complete Σ-consistent set Δ, there is a world in the canonical model where all and only the formulas in Δ are true. So Lindenbaum's Lemma guarantees that every Σ-consistent set is true at some world in the canonical model.

mod:com:lin: thm:lindenbaum

Theorem com.1 (Lindenbaum's Lemma)**.** *If Γ is Σ-consistent then there is a complete Σ-consistent set Δ extending Γ.*

Proof. Let φ_0, φ_1, ... be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing p_0, and at each stage $n \geq 1$ list the finitely many formulas of length n using only variables among $p_0, \dots, p_n$. We define sets of formulas Δ_n by induction on n, and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$. Supposing that Δ_n has been defined, we define Δ_{n+1} by:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

If we now let $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$.

We have to show that this definition actually yields a set Δ with the required properties, i.e., $\Gamma \subseteq \Delta$ and Δ is complete Σ-consistent.

It's obvious that $\Gamma \subseteq \Delta$, since $\Delta_0 \subseteq \Delta$ by construction, and $\Delta_0 = \Gamma$. In fact, $\Delta_n \subseteq \Delta$ for all n, since Δ is the union of all Δ_n. (Since in each step of the construction, we add a formula to the set already constructed, $\Delta_n \subseteq \Delta_{n+1}$, so since $\subseteq$ is transitive, $\Delta_n \subseteq \Delta_m$ whenever $n \leq m$.) At each stage of the construction, we either add φ_n or $\neg\varphi_n$, and every formula appears (at least once) in the list of all φ_n. So, for every φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$, so Δ is complete by definition.

Finally, we have to show, that Δ is Σ-consistent. To do this, we show that (a) if Δ were Σ-inconsistent, then some Δ_n would be Σ-inconsistent, and (b) all Δ_n are Σ-consistent.

So suppose Δ were Σ-inconsistent. Then $\Delta \vdash_\Sigma \bot$, i.e., there are φ_1, ..., $\varphi_k \in \Delta$ such that $\Sigma \vdash \varphi_1 \to (\varphi_2 \to \cdots (\varphi_k \to \bot) \dots)$. Since $\Delta = \bigcap_{n=0}^{\infty}$, each $\varphi_i \in \Delta_{n_i}$ for some n_i. Let n be the largest of these. Since $n_i \leq n$, $\Delta_{n_i} \subseteq \Delta_n$. So, all φ_i are in some Δ_n. This would mean $\Delta_n \vdash_\Sigma \bot$, i.e., Δ_n is Σ-inconsistent.

To show that each Δ_n is Σ-consistent, we use a simple induction on n. $\Delta_0 = \Gamma$, and we assumed Γ was Σ-consistent. So the claim holds for $n = 0$. Now suppose it holds for n, i.e., Δ_n is Σ-consistent. Δ_{n+1} is either $\Delta_n \cup \{\varphi_n\}$ is that is Σ-consistent, otherwise it is $\Delta_n \cup \{\neg\varphi_n\}$. In the first case, Δ_{n+1} is clearly Σ-consistent. However, by **????**, either $\Delta_n \cup \{\varphi_n\}$ or $\Delta_n \cup \{\neg\varphi_n\}$ is consistent, so Δ_{n+1} is consistent in the other case as well. □

Corollary com.2. *$\Gamma \vdash_\Sigma \varphi$ if and only if $\varphi \in \Delta$ for each complete Σ-consistent set Δ extending Γ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system Σ.)* mod:com:lin: cor:provability-characterization

Proof. Suppose $\Gamma \vdash_\Sigma \varphi$, and let Δ be any complete Σ-consistent set extending Γ. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_\Sigma \varphi$ (by monotony) and $\Delta \vdash_\Sigma \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \nvdash_\Sigma \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent, and by Lindenbaum's Lemma there is a complete consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$. By consistency, $\varphi \notin \Delta$. □

Photo Credits

Bibliography