Lindenbaum’s Lemma establishes that every $\Sigma$-consistent set of formulas is contained in at least one complete $\Sigma$-consistent set. Our construction of the canonical model will show that for each complete $\Sigma$-consistent set $\Delta$, there is a world in the canonical model where all and only the formulas in $\Delta$ are true. So Lindenbaum’s Lemma guarantees that every $\Sigma$-consistent set is true at some world in the canonical model.

**Theorem com.1 (Lindenbaum’s Lemma).** If $\Gamma$ is $\Sigma$-consistent then there is a complete $\Sigma$-consistent set $\Delta$ extending $\Gamma$.

**Proof.** Let $\varphi_0, \varphi_1, \ldots$ be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing $p_0$, and at each stage $n \geq 1$ list the finitely many formulas of length $n$ using only variables among $p_0, \ldots, p_n$. We define sets of formulas $\Delta_n$ by induction on $n$, and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$. Supposing that $\Delta_n$ has been defined, we define $\Delta_{n+1}$ by:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{ \varphi_n \}, & \text{if } \Delta_n \cup \{ \varphi_n \} \text{ is consistent;} \\ \Delta_n \cup \{ \neg \varphi_n \}, & \text{otherwise.} \end{cases}$$

If we now let $\Delta = \bigcup_{n=0}^\infty \Delta_n$.

We have to show that this definition actually yields a set $\Delta$ with the required properties, i.e., $\Gamma \subseteq \Delta$ and $\Delta$ is complete $\Sigma$-consistent.

It’s obvious that $\Gamma \subseteq \Delta$, since $\Delta_0 \subseteq \Delta$ by construction, and $\Delta_0 = \Gamma$. In fact, $\Delta_n \subseteq \Delta$ for all $n$, since $\Delta$ is the union of all $\Delta_n$. (Since in each step of the construction, we add a formula to the set already constructed, $\Delta_n \subseteq \Delta_{n+1}$, so since $\subseteq$ is transitive, $\Delta_n \subseteq \Delta_m$ whenever $n \leq m$.) At each stage of the construction, we either add $\varphi_n$ or $\neg \varphi_n$, and every formula appears (at least once) in the list of all $\varphi_n$. So, for every $\varphi$ either $\varphi \in \Delta$ or $\neg \varphi \in \Delta$, so $\Delta$ is complete by definition.

Finally, we have to show, that $\Delta$ is $\Sigma$-consistent. To do this, we show that (a) if $\Delta$ were $\Sigma$-inconsistent, then some $\Delta_n$ would be $\Sigma$-inconsistent, and (b) all $\Delta_n$ are $\Sigma$-consistent.

So suppose $\Delta$ were $\Sigma$-inconsistent. Then $\Delta \vdash_\Sigma \bot$, i.e., there are $\varphi_1, \ldots, \varphi_k \in \Delta$ such that $\Sigma \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_k \rightarrow \bot) \ldots)$. Since $\Delta = \bigcap_{n=0}^\infty$, each $\varphi_i \in \Delta_n$ for some $n_i$. Let $n$ be the largest of these. Since $n_i \leq n$, $\Delta_n \subseteq \Delta_n$. So, all $\varphi_i$ are in some $\Delta_n$. This would mean $\Delta_n \vdash_\Sigma \bot$, i.e., $\Delta_n$ is $\Sigma$-inconsistent.

To show that each $\Delta_n$ is $\Sigma$-consistent, we use a simple induction on $n$. $\Delta_0 = \Gamma$, and we assumed $\Gamma$ was $\Sigma$-consistent. So the claim holds for $n = 0$. Now suppose it holds for $n$, i.e., $\Delta_n$ is $\Sigma$-consistent. $\Delta_{n+1}$ is either $\Delta_n \cup \{ \varphi_n \}$ or $\Delta_n \cup \{ \neg \varphi_n \}$, and it is either $\Sigma$-consistent, otherwise it is $\Sigma$-inconsistent. In the first case, $\Delta_{n+1}$ is clearly $\Sigma$-consistent. However, by ????, either $\Delta_n \cup \{ \varphi_n \}$ or $\Delta_n \cup \{ \neg \varphi_n \}$ is consistent, so $\Delta_{n+1}$ is consistent in the other case as well.
Corollary com.2. $\Gamma \vdash_\Sigma \varphi$ if and only if $\varphi \in \Delta$ for each complete $\Sigma$-consistent set $\Delta$ extending $\Gamma$ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system $\Sigma$.)

Proof. Suppose $\Gamma \vdash_\Sigma \varphi$, and let $\Delta$ be any complete $\Sigma$-consistent set extending $\Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg \varphi \in \Delta$ and so $\Delta \vdash_\Sigma \varphi$ (by monotony) and $\Delta \vdash_\Sigma \neg \varphi$ (by reflexivity), and so $\Delta$ is inconsistent. Conversely if $\Gamma \nvdash_\Sigma \varphi$, then $\Gamma \cup \{\neg \varphi\}$ is $\Sigma$-consistent, and by Lindenbaum’s Lemma there is a complete consistent set $\Delta$ extending $\Gamma \cup \{\neg \varphi\}$. By consistency, $\varphi \notin \Delta$. \qed

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Bibliography