com.1 Lindenbaum's Lemma

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mod:com:lin:thm:lindenbaum **Theorem com.1** (Lindenbaum's Lemma). *If Γ is Σ-consistent then there is a complete Σ-consistent set Δ extending Γ.*

Proof. Let φ_0, φ_1, ... be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing p_0, and at each stage n list the finitely many formulas of length n using only variables among p_0, ..., p_n. We define sets of formulas Δ_n by induction on n, and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$, then supposing that Δ_n has been defined:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

If we now let $\Delta = \bigcup_n \Delta_n$, we can show the following:

1. For each n, $\Delta_n \subseteq \Delta$ (immediate from the definition).
2. $\Gamma \subseteq \Delta$ (from (a)).
3. If $n \leq m$ then $\Delta_n \subseteq \Delta_m$ (by induction on $m - n$).
4. Δ is maximal (by construction).
5. For each m, Δ_m is consistent (by induction on m, using **????**)
6. If $\Delta' \subseteq \Delta$ is finite, then there is m such that $\Delta' \subseteq \Delta_m$.
7. Δ is consistent.

It follows that Δ is a complete Σ-consistent set extending Γ. □

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Bibliography