

com.1 Lindenbaum's Lemma

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Lindenbaum's Lemma establishes that every Σ -consistent set of formulas is contained in at least one complete Σ -consistent set. Our construction of the canonical model will show that for each complete Σ -consistent set Δ , there is a world in the canonical model where all and only the formulas in Δ are true. So Lindenbaum's Lemma guarantees that every Σ -consistent set is true at some world in the canonical model.

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Theorem com.1 (Lindenbaum's Lemma). *If Γ is Σ -consistent then there is a complete Σ -consistent set Δ extending Γ .*

Proof. Let $\varphi_0, \varphi_1, \dots$ be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing ρ_0 , and at each stage $n \geq 1$ list the finitely many formulas of length n using only variables among ρ_0, \dots, ρ_n . We define sets of formulas Δ_n by induction on n , and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$. Supposing that Δ_n has been defined, we define Δ_{n+1} by:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is } \Sigma\text{-consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

Now let $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$.

We have to show that this definition actually yields a set Δ with the required properties, i.e., $\Gamma \subseteq \Delta$ and Δ is complete Σ -consistent.

It's obvious that $\Gamma \subseteq \Delta$, since $\Delta_0 \subseteq \Delta$ by construction, and $\Delta_0 = \Gamma$. In fact, $\Delta_n \subseteq \Delta$ for all n , since Δ is the union of all Δ_n . (Since in each step of the construction, we add a formula to the set already constructed, $\Delta_n \subseteq \Delta_{n+1}$, so since \subseteq is transitive, $\Delta_n \subseteq \Delta_m$ whenever $n \leq m$.) At each stage of the construction, we either add φ_n or $\neg\varphi_n$, and every formula appears (at least once) in the list of all φ_n . So, for every φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$, so Δ is complete by definition.

Finally, we have to show, that Δ is Σ -consistent. To do this, we show that (a) if Δ were Σ -inconsistent, then some Δ_n would be Σ -inconsistent, and (b) all Δ_n are Σ -consistent.

So suppose Δ were Σ -inconsistent. Then $\Delta \vdash_{\Sigma} \perp$, i.e., there are $\varphi_1, \dots, \varphi_k \in \Delta$ such that $\Sigma \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_k \rightarrow \perp) \dots)$. Since $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$, each $\varphi_i \in \Delta_{n_i}$ for some n_i . Let n be the largest of these. Since $n_i \leq n$, $\Delta_{n_i} \subseteq \Delta_n$. So, all φ_i are in some Δ_n . This would mean $\Delta_n \vdash_{\Sigma} \perp$, i.e., Δ_n is Σ -inconsistent.

To show that each Δ_n is Σ -consistent, we use a simple induction on n . $\Delta_0 = \Gamma$, and we assumed Γ was Σ -consistent. So the claim holds for $n = 0$. Now suppose it holds for n , i.e., Δ_n is Σ -consistent. Δ_{n+1} is either $\Delta_n \cup \{\varphi_n\}$ if that is Σ -consistent, otherwise it is $\Delta_n \cup \{\neg\varphi_n\}$. In the first case, Δ_{n+1} is clearly Σ -consistent. However, by ????, either $\Delta_n \cup \{\varphi_n\}$ or $\Delta_n \cup \{\neg\varphi_n\}$ is consistent, so Δ_{n+1} is consistent in the other case as well. \square

Corollary com.2. $\Gamma \vdash_{\Sigma} \varphi$ if and only if $\varphi \in \Delta$ for each complete Σ -consistent set Δ extending Γ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system Σ .) [nml:com:lin:](#)
[cor:provability-characterization](#)

Proof. Suppose $\Gamma \vdash_{\Sigma} \varphi$, and let Δ be any complete Σ -consistent set extending Γ . If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_{\Sigma} \varphi$ (by monotonicity) and $\Delta \vdash_{\Sigma} \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \not\vdash_{\Sigma} \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ -consistent, and by Lindenbaum's Lemma there is a complete consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$. By consistency, $\varphi \notin \Delta$. \square

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Bibliography