

com.1 Introduction

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If Σ is a modal system, then the soundness theorem establishes that if $\Sigma \vdash \varphi$, then φ is valid in any class \mathcal{C} of models in which all instances of all **formulas** in Σ are valid. In particular that means that if $\mathbf{K} \vdash \varphi$ then φ is true in all models; if $\mathbf{KT} \vdash \varphi$ then φ is true in all reflexive models; if $\mathbf{KD} \vdash \varphi$ then φ is true in all serial models, etc.

Completeness is the converse of soundness: that \mathbf{K} is complete means that if a **formula** φ is valid, $\vdash \varphi$, for instance. Proving completeness is a lot harder to do than proving soundness. It is useful, first, to consider the contrapositive: \mathbf{K} is complete iff whenever $\not\vdash \varphi$, there is a countermodel, i.e., a model \mathfrak{M} such that $\mathfrak{M} \not\models \varphi$. Equivalently (negating φ), we could prove that whenever $\not\vdash \neg\varphi$, there is a model of φ . In the construction of such a model, we can use information contained in φ . When we find models for specific **formulas** we often do the same: E.g., if we want to find a countermodel to $p \rightarrow \Box q$, we know that it has to contain a world where p is true and $\Box q$ is false. And a world where $\Box q$ is false means there has to be a world accessible from it where q is false. And that's all we need to know: which worlds make the **propositional variables** true, and which worlds are accessible from which worlds.

In the case of proving completeness, however, we don't have a specific **formula** φ for which we are constructing a model. We want to establish that a model exists for every φ such that $\not\vdash_{\Sigma} \neg\varphi$. This is a minimal requirement, since if $\vdash_{\Sigma} \neg\varphi$, by soundness, there is no model for φ (in which Σ is true). Now note that $\not\vdash_{\Sigma} \neg\varphi$ iff φ is Σ -consistent. (Recall that $\Sigma \not\vdash_{\Sigma} \neg\varphi$ and $\varphi \not\vdash_{\Sigma} \perp$ are equivalent.) So our task is to construct a model for every Σ -consistent **formula**.

The trick we'll use is to find a Σ -consistent set of **formulas** that contains φ , but also other formulas which tell us what the world that makes φ true has to look like. Such sets are *complete* Σ -consistent sets. But it's not enough to construct a models with a single world to make φ true, it will have to contain many world and an accessibility relation. The complete Σ -consistent set containing φ will also contain other **formulas** of the form $\Box\psi$ and $\Diamond\chi$. In all accessible worlds, ψ has to be true; in at least one, χ has to be true. In order to accomplish this, we'll simply take *all* possible complete Σ -consistent sets as the basis for the set of worlds. A tricky part will be to figure out when a complete Σ -consistent set should count as being accessible from another in our model.

We'll show that in the model so defined, φ is true at a world—which is also a complete Σ -consistent set—iff φ is an **element** of that set. If φ is Σ -consistent, it will be an **element** of at least one complete Σ -consistent set (a fact we'll prove), and so there will be a world where φ is true. So we will have a single model where every Σ -consistent **formula** φ is true at some world. This single model is the canonical model for Σ .

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Bibliography