If $\Sigma$ is a modal system, then the soundness theorem establishes that if $\Sigma \vdash \varphi$, then $\varphi$ is valid in any class $\mathcal{C}$ of models in which all instances of all formulas in $\Sigma$ are valid. In particular that means that if $K \vdash \varphi$ then $\varphi$ is true in all models; if $KT \vdash \varphi$ then $\varphi$ is true in all reflexive models; if $KD \vdash \varphi$ then $\varphi$ is true in all serial models, etc.

Completeness is the converse of soundness: that $K$ is complete means that if a formula $\varphi$ is valid, $\vdash \varphi$, for instance. Proving completeness is a lot harder to do than proving soundness. It is useful, first, to consider the contrapositive: $K$ is complete iff whenever $\not\vdash \varphi$, there is a countermodel, i.e., a model $M$ such that $M \not\models \varphi$. Equivalently (negating $\varphi$), we could prove that whenever $\not\vdash \neg \varphi$, there is a model of $\varphi$. In the construction of such a model, we can use information contained in $\varphi$. When we find models for specific formulas we often do the same: E.g., if we want to find a countermodel to $p \to \Box q$, we know that it has to contain a world where $p$ is true and $\Box q$ is false. And a world where $\Box q$ is false means there has to be a world accessible from it where $q$ is false. And that’s all we need to know: which worlds make the propositional variables true, and which worlds are accessible from which worlds.

In the case of proving completeness, however, we don’t have a specific formula $\varphi$ for which we are constructing a model. We want to establish that a model exists for every $\varphi$ such that $\not\vdash \Sigma \neg \varphi$. This is a minimal requirement, since if $\vdash \Sigma \neg \varphi$, by soundness, there is no model for $\varphi$ (in which $\Sigma$ is true). Now note that $\not\vdash \Sigma \neg \varphi$ iff $\varphi$ is $\Sigma$-consistent. (Recall that $\Sigma \not\vdash \neg \varphi$ and $\varphi \not\vdash \Sigma \bot$ are equivalent.) So our task is to construct a model for every $\Sigma$-consistent formula.

The trick we’ll use is to find a $\Sigma$-consistent set of formulas that contains $\varphi$, but also other formulas which tell us what the world that makes $\varphi$ true has to look like. Such sets are complete $\Sigma$-consistent sets. It’s not enough to construct a model with a single world to make $\varphi$ true, it will have to contain multiple worlds and an accessibility relation. The complete $\Sigma$-consistent set containing $\varphi$ will also contain other formulas of the form $\Box \psi$ and $\Diamond \chi$. In all accessible worlds, $\psi$ has to be true; in at least one, $\chi$ has to be true. In order to accomplish this, we’ll simply take all possible complete $\Sigma$-consistent sets as the basis for the set of worlds. A tricky part will be to figure out when a complete $\Sigma$-consistent set should count as being accessible from another in our model.

We’ll show that in the model so defined, $\varphi$ is true at a world—which is also a complete $\Sigma$-consistent set—iff $\varphi$ is an element of that set. If $\varphi$ is $\Sigma$-consistent, it will be an element of at least one complete $\Sigma$-consistent set (a fact we’ll prove), and so there will be a world where $\varphi$ is true. So we will have a single model where every $\Sigma$-consistent formula $\varphi$ is true at some world. This single model is the canonical model for $\Sigma$. 