

## com.1 Introduction

mod:com:int:  
sec

If  $\Sigma$  is a modal system, then the soundness theorem establishes that if  $\Sigma \vdash \varphi$ , then  $\varphi$  is valid in any class  $\mathcal{C}$  of models in which all instances of all **formulas** in  $\Sigma$  are valid. In particular that means that if  $\mathbf{K} \vdash \varphi$  then  $\varphi$  is true in all models; if  $\mathbf{KT} \vdash \varphi$  then  $\varphi$  is true in all reflexive models; if  $\mathbf{KD} \vdash \varphi$  then  $\varphi$  is true in all serial models, etc.

Completeness is the converse of soundness: that  $\mathbf{K}$  is complete means that if a **formula**  $\varphi$  is valid,  $\vdash \varphi$ , for instance. Proving completeness is a lot harder to do than proving soundness. It is useful, first, to consider the contrapositive:  $\mathbf{K}$  is complete iff whenever  $\not\vdash \varphi$ , there is a countermodel, i.e., a model  $\mathfrak{M}$  such that  $\mathfrak{M} \not\models \varphi$ . Equivalently (negating  $\varphi$ ), we could prove that whenever  $\not\vdash \neg\varphi$ , there is a model of  $\varphi$ . In the construction of such a model, we can use information contained in  $\varphi$ . When we find models for specific **formulas** we often do the same: E.g., if we want to find a countermodel to  $p \rightarrow \Box q$ , we know that it has to contain a world where  $p$  is true and  $\Box q$  is false. And a world where  $\Box q$  is false means there has to be a world accessible from it where  $q$  is false. And that's all we need to know: which worlds make the **propositional variables** true, and which worlds are accessible from which worlds.

In the case of proving completeness, however, we don't have a specific **formula**  $\varphi$  for which we are constructing a model. We want to establish that a model exists for every  $\varphi$  such that  $\not\vdash_{\Sigma} \neg\varphi$ . This is a minimal requirement, since if  $\vdash_{\Sigma} \neg\varphi$ , by soundness, there is no model for  $\varphi$  (in which  $\Sigma$  is true). Now note that  $\not\vdash_{\Sigma} \neg\varphi$  iff  $\varphi$  is  $\Sigma$ -consistent. (Recall that  $\Sigma \not\vdash_{\Sigma} \neg\varphi$  and  $\varphi \not\vdash_{\Sigma} \perp$  are equivalent.) So our task is to construct a model for every  $\Sigma$ -consistent **formula**.

The trick we'll use is to find a  $\Sigma$ -consistent set of **formulas** that contains  $\varphi$ , but also other formulas which tell us what the world that makes  $\varphi$  true has to look like. Such sets are *complete*  $\Sigma$ -consistent sets. It's not enough to construct a model with a single world to make  $\varphi$  true, it will have to contain multiple worlds and an accessibility relation. The complete  $\Sigma$ -consistent set containing  $\varphi$  will also contain other **formulas** of the form  $\Box\psi$  and  $\Diamond\chi$ . In all accessible worlds,  $\psi$  has to be true; in at least one,  $\chi$  has to be true. In order to accomplish this, we'll simply take *all* possible complete  $\Sigma$ -consistent sets as the basis for the set of worlds. A tricky part will be to figure out when a complete  $\Sigma$ -consistent set should count as being accessible from another in our model.

We'll show that in the model so defined,  $\varphi$  is true at a world—which is also a complete  $\Sigma$ -consistent set—iff  $\varphi$  is an **element** of that set. If  $\varphi$  is  $\Sigma$ -consistent, it will be an **element** of at least one complete  $\Sigma$ -consistent set (a fact we'll prove), and so there will be a world where  $\varphi$  is true. So we will have a single model where every  $\Sigma$ -consistent **formula**  $\varphi$  is true at some world. This single model is the *canonical* model for  $\Sigma$ .

**Photo Credits**

**Bibliography**