

com.1 Frame Completeness

nml:com:fra:sec The completeness theorem for **K** can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

nml:com:fra:thm:completeframeprops **Theorem com.1.** *If a normal modal logic Σ contains one of the formulas on the left-hand side of Table 1, then the canonical model for Σ has the corresponding property on the right-hand side.*

If Σ contains the canonical model for Σ is:
D: $\Box\varphi \rightarrow \Diamond\varphi$	serial;
T: $\Box\varphi \rightarrow \varphi$	reflexive;
B: $\varphi \rightarrow \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \rightarrow \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$	euclidean.

Table 1: Basic correspondence facts.

nml:com:fra:tab:correspondencetable

Proof. We take each of these up in turn.

Suppose Σ contains D, and let $\Delta \in W^\Sigma$; we need to show that there is a Δ' such that $R^\Sigma \Delta \Delta'$. It suffices to show that $\Box^{-1}\Delta$ is Σ -consistent, for then by Lindenbaum's Lemma, there is a complete Σ -consistent set $\Delta' \supseteq \Box^{-1}\Delta$, and by definition of R^Σ we have $R^\Sigma \Delta \Delta'$. So, suppose for contradiction that $\Box^{-1}\Delta$ is *not* Σ -consistent, i.e., $\Box^{-1}\Delta \vdash_\Sigma \perp$. By ??, $\Delta \vdash_\Sigma \Box\perp$, and since Σ contains D, also $\Delta \vdash_\Sigma \Diamond\perp$. But Σ is normal, so $\Sigma \vdash \neg\Diamond\perp$ (??), whence also $\Delta \vdash_\Sigma \neg\Diamond\perp$, against the consistency of Δ .

Now suppose Σ contains T, and let $\Delta \in W^\Sigma$. We want to show $R^\Sigma \Delta \Delta$, i.e., $\Box^{-1}\Delta \subseteq \Delta$. But if $\Box\varphi \in \Delta$ then by T also $\varphi \in \Delta$, as desired.

Now suppose Σ contains B, and suppose $R^\Sigma \Delta \Delta'$ for $\Delta, \Delta' \in W^\Sigma$. We need to show that $R^\Sigma \Delta' \Delta$, i.e., $\Box^{-1}\Delta' \subseteq \Delta$. By ??, this is equivalent to $\Diamond\Delta \subseteq \Delta'$. So suppose $\varphi \in \Delta$. By B, also $\Box\Diamond\varphi \in \Delta$. By the hypothesis that $R^\Sigma \Delta \Delta'$, we have that $\Box^{-1}\Delta \subseteq \Delta'$, and hence $\Diamond\varphi \in \Delta'$, as required.

Now suppose Σ contains 4, and suppose $R^\Sigma \Delta_1 \Delta_2$ and $R^\Sigma \Delta_2 \Delta_3$. We need to show $R^\Sigma \Delta_1 \Delta_3$. From the hypothesis we have both $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and $\Box^{-1}\Delta_2 \subseteq \Delta_3$. In order to show $R^\Sigma \Delta_1 \Delta_3$ it suffices to show $\Box^{-1}\Delta_1 \subseteq \Delta_3$. So let $\psi \in \Box^{-1}\Delta_1$, i.e., $\Box\psi \in \Delta_1$. By 4, also $\Box\Box\psi \in \Delta_1$ and by hypothesis we get, first, that $\Box\psi \in \Delta_2$ and, second, that $\psi \in \Delta_3$, as desired.

Now suppose Σ contains 5, suppose $R^\Sigma \Delta_1 \Delta_2$ and $R^\Sigma \Delta_1 \Delta_3$. We need to show $R^\Sigma \Delta_2 \Delta_3$. The first hypothesis gives $\Box^{-1}\Delta_1 \subseteq \Delta_2$, and the second hypothesis is equivalent to $\Diamond\Delta_3 \subseteq \Delta_2$, by ??. To show $R^\Sigma \Delta_2 \Delta_3$, by ??, it suffices to show $\Diamond\Delta_3 \subseteq \Delta_2$. So let $\Diamond\varphi \in \Diamond\Delta_3$, i.e., $\varphi \in \Delta_3$. By the second hypothesis $\Diamond\varphi \in \Delta_1$ and by 5, $\Box\Diamond\varphi \in \Delta_1$ as well. But now the first hypothesis gives $\Diamond\varphi \in \Delta_2$, as desired. \square

As a corollary we obtain completeness results for a number of systems. For instance, we know that **S5** = **KT5** = **KTB4** is complete with respect to the

class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

Theorem com.2. *Let \mathcal{C}_D , \mathcal{C}_T , \mathcal{C}_B , \mathcal{C}_4 , and \mathcal{C}_5 be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1, \dots, \varphi_n$ among D, T, B, 4, and 5, the system $\mathbf{K}\varphi_1 \dots \varphi_n$ is determined by the class of models $\mathcal{C} = \mathcal{C}_{\varphi_1} \cap \dots \cap \mathcal{C}_{\varphi_n}$.* nml:com:fra: thm:generaldet

Proposition com.3. *Let Σ be a normal modal logic; then:*

1. *If Σ contains the schema $\diamond\varphi \rightarrow \Box\varphi$ then the canonical model for Σ is partially functional.* nml:com:fra: prop:anotherfive-a
2. *If Σ contains the schema $\diamond\varphi \leftrightarrow \Box\varphi$ then the canonical model for Σ is functional.*
3. *If Σ contains the schema $\Box\Box\varphi \rightarrow \Box\varphi$ then the canonical model for Σ is weakly dense.*

(see ?? for definitions of these frame properties).

Proof. 1. Suppose that Σ contains the schema $\diamond\varphi \rightarrow \Box\varphi$, to show that R^Σ is partially functional we need to prove that for any $\Delta_1, \Delta_2, \Delta_3 \in W^\Sigma$, if $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$ then $\Delta_2 = \Delta_3$. Since $R^\Sigma\Delta_1\Delta_2$ we have $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and since $R^\Sigma\Delta_1\Delta_3$ also $\Box^{-1}\Delta_1 \subseteq \Delta_3$. The identity $\Delta_2 = \Delta_3$ will follow if we can establish the two inclusions $\Delta_2 \subseteq \Delta_3$ and $\Delta_3 \subseteq \Delta_2$. For the first inclusion, let $\varphi \in \Delta_2$; then $\diamond\varphi \in \Delta_1$, and by the schema and deductive closure of Δ_1 also $\Box\varphi \in \Delta_1$, whence by the hypothesis that $R^\Sigma\Delta_1\Delta_3$, $\varphi \in \Delta_3$. The second inclusion is similar.

2. This follows immediately from part (1) and the seriality proof in [Theorem com.1](#).
3. Suppose Σ contains the schema $\Box\Box\varphi \rightarrow \Box\varphi$ and to show that R^Σ is weakly dense, let $R^\Sigma\Delta_1\Delta_2$. We need to show that there is a complete Σ -consistent set Δ_3 such that $R^\Sigma\Delta_1\Delta_3$ and $R^\Sigma\Delta_3\Delta_2$. Let:

$$\Gamma = \Box^{-1}\Delta_1 \cup \diamond\Delta_2.$$

It suffices to show that Γ is Σ -consistent, for then by Lindenbaum's Lemma it can be extended to a complete Σ -consistent set Δ_3 such that $\Box^{-1}\Delta_1 \subseteq \Delta_3$ and $\diamond\Delta_2 \subseteq \Delta_3$, i.e., $R^\Sigma\Delta_1\Delta_3$ and $R^\Sigma\Delta_3\Delta_2$ (by ??).

Suppose for contradiction that Γ is not consistent. Then there are formulas $\Box\varphi_1, \dots, \Box\varphi_n \in \Delta_1$ and $\psi_1, \dots, \psi_m \in \Delta_2$ such that

$$\varphi_1, \dots, \varphi_n, \diamond\psi_1, \dots, \diamond\psi_m \vdash_\Sigma \perp.$$

Since $\Diamond(\psi_1 \wedge \dots \wedge \psi_m) \rightarrow (\Diamond\psi_1 \wedge \dots \wedge \Diamond\psi_m)$ is **derivable** in every normal modal logic, we argue as follows, contradicting the consistency of Δ_2 :

$$\begin{aligned}
& \varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_{\Sigma} \perp \\
& \varphi_1, \dots, \varphi_n \vdash_{\Sigma} (\Diamond\psi_1 \wedge \dots \wedge \Diamond\psi_m) \rightarrow \perp \\
& \quad \text{by the deduction theorem} \\
& \quad \text{????, and TAUT} \\
& \varphi_1, \dots, \varphi_n \vdash_{\Sigma} \Diamond(\psi_1 \wedge \dots \wedge \psi_m) \rightarrow \perp \\
& \quad \text{since } \Sigma \text{ is normal} \\
& \varphi_1, \dots, \varphi_n \vdash_{\Sigma} \neg\Diamond(\psi_1 \wedge \dots \wedge \psi_m) \\
& \quad \text{by PL} \\
& \varphi_1, \dots, \varphi_n \vdash_{\Sigma} \Box\neg(\psi_1 \wedge \dots \wedge \psi_m) \\
& \quad \Box\neg \text{ for } \neg\Diamond \\
& \Box\varphi_1, \dots, \Box\varphi_n \vdash_{\Sigma} \Box\Box\neg(\psi_1 \wedge \dots \wedge \psi_m) \\
& \quad \text{by ??} \\
& \Box\varphi_1, \dots, \Box\varphi_n \vdash_{\Sigma} \Box\neg(\psi_1 \wedge \dots \wedge \psi_m) \\
& \quad \text{by schema } \Box\Box\varphi \rightarrow \Box\varphi \\
& \Delta_1 \vdash_{\Sigma} \Box\neg(\psi_1 \wedge \dots \wedge \psi_m) \\
& \quad \text{by monotonicity, ????} \\
& \Box\neg(\psi_1 \wedge \dots \wedge \psi_m) \in \Delta_1 \\
& \quad \text{by deductive closure;} \\
& \neg(\psi_1 \wedge \dots \wedge \psi_m) \in \Delta_2 \\
& \quad \text{since } R^{\Sigma} \Delta_1 \Delta_2. \quad \square
\end{aligned}$$

On the strength of these examples, one might think that every system Σ of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of Σ is valid. Unfortunately, there are many systems that are not complete in this sense.

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Bibliography