

## com.1 Frame Completeness

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sec

The completeness theorem for **K** can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

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thm:completeframeprops

**Theorem com.1.** *If a normal modal logic  $\Sigma$  contains one of the formulas on the left-hand side of table 1, then the canonical model for  $\Sigma$  has the corresponding property on the right-hand side.*

If $\Sigma$ contains ...	... the canonical model for $\Sigma$ is:
D: $\Box\varphi \rightarrow \Diamond\varphi$	serial;
T: $\Box\varphi \rightarrow \varphi$	reflexive;
B: $\varphi \rightarrow \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \rightarrow \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$	euclidean.

Table 1: Basic correspondence facts.

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tab:correspondencetable

*Proof.* We take each of these up in turn.

Suppose  $\Sigma$  contains D, and let  $\Delta \in W^\Sigma$ ; we need to show that there is a  $\Delta'$  such that  $R^\Sigma \Delta \Delta'$ . It suffices to show that  $\Box^{-1}\Delta$  is  $\Sigma$ -consistent, for then by Lindenbaum's Lemma, there is a complete  $\Sigma$ -consistent set  $\Delta' \supseteq \Box^{-1}\Delta$ , and by definition of  $R^\Sigma$  we have  $R^\Sigma \Delta \Delta'$ . So, suppose for contradiction that  $\Box^{-1}\Delta$  is *not*  $\Sigma$ -consistent, i.e.,  $\Box^{-1}\Delta \vdash_\Sigma \perp$ . By ??,  $\Delta \vdash_\Sigma \Box\perp$ , and since  $\Sigma$  contains D, also  $\Delta \vdash_\Sigma \Diamond\perp$ . But  $\Sigma$  is normal, so  $\Sigma \vdash \neg\Diamond\perp$  (??), whence also  $\Delta \vdash_\Sigma \neg\Diamond\perp$ , against the consistency of  $\Delta$ .

Now suppose  $\Sigma$  contains T, and let  $\Delta \in W^\Sigma$ . We want to show  $R^\Sigma \Delta \Delta$ , i.e.,  $\Box^{-1}\Delta \subseteq \Delta$ . But if  $\Box\varphi \in \Delta$  then by T also  $\varphi \in \Delta$ , as desired.

Now suppose  $\Sigma$  contains B, and suppose  $R^\Sigma \Delta \Delta'$  for  $\Delta, \Delta' \in W^\Sigma$ . We need to show that  $R^\Sigma \Delta' \Delta$ , i.e.,  $\Box^{-1}\Delta' \subseteq \Delta$ . By ??, this is equivalent to  $\Diamond\Delta \subseteq \Delta'$ . So suppose  $\varphi \in \Delta$ . By B, also  $\Box\Diamond\varphi \in \Delta$ . By the hypothesis that  $R^\Sigma \Delta \Delta'$ , we have that  $\Box^{-1}\Delta \subseteq \Delta'$ , and hence  $\Diamond\varphi \in \Delta'$ , as required.

Now suppose  $\Sigma$  contains 4, and suppose  $R^\Sigma \Delta_1 \Delta_2$  and  $R^\Sigma \Delta_2 \Delta_3$ . We need to show  $R^\Sigma \Delta_1 \Delta_3$ . From the hypothesis we have both  $\Box^{-1}\Delta_1 \subseteq \Delta_2$  and  $\Box^{-1}\Delta_2 \subseteq \Delta_3$ . In order to show  $R^\Sigma \Delta_1 \Delta_3$  it suffices to show  $\Box^{-1}\Delta_1 \subseteq \Delta_3$ . So let  $\psi \in \Box^{-1}\Delta_1$ , i.e.,  $\Box\psi \in \Delta_1$ . By 4, also  $\Box\Box\psi \in \Delta_1$  and by hypothesis we get, first, that  $\Box\psi \in \Delta_2$  and, second, that  $\psi \in \Delta_3$ , as desired.

Now suppose  $\Sigma$  contains 5, suppose  $R^\Sigma \Delta_1 \Delta_2$  and  $R^\Sigma \Delta_1 \Delta_3$ . We need to show  $R^\Sigma \Delta_2 \Delta_3$ . The first hypothesis gives  $\Box^{-1}\Delta_1 \subseteq \Delta_2$ , and the second hypothesis is equivalent to  $\Diamond\Delta_3 \subseteq \Delta_2$ , by ??. To show  $R^\Sigma \Delta_2 \Delta_3$ , by ??, it suffices to show  $\Diamond\Delta_3 \subseteq \Delta_2$ . So let  $\Diamond\varphi \in \Diamond\Delta_3$ , i.e.,  $\varphi \in \Delta_3$ . By the second hypothesis  $\Diamond\varphi \in \Delta_1$  and by 5,  $\Box\Diamond\varphi \in \Delta_1$  as well. But now the first hypothesis gives  $\Diamond\varphi \in \Delta_2$ , as desired.  $\square$

As a corollary we obtain completeness results for a number of systems. For instance, we know that **S5** = **KT5** = **KTB4** is complete with respect to the

class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

**Theorem com.2.** *Let  $\mathcal{C}_D$ ,  $\mathcal{C}_T$ ,  $\mathcal{C}_B$ ,  $\mathcal{C}_4$ , and  $\mathcal{C}_5$  be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas  $\varphi_1, \dots, \varphi_n$  among D, T, B, 4, and 5, the system  $\mathbf{K}\varphi_1 \dots \varphi_n$  is determined by the class of models  $\mathcal{C} = \mathcal{C}_{\varphi_1} \cap \dots \cap \mathcal{C}_{\varphi_n}$ .* *mod:com:fra: thm:generaldet*

**Proposition com.3.** *Let  $\Sigma$  be a normal modal logic; then:*

1. *If  $\Sigma$  contains the schema  $\diamond\varphi \rightarrow \Box\varphi$  then the canonical model for  $\Sigma$  is partially functional.* *mod:com:fra: prop:anotherfive-a*
2. *If  $\Sigma$  contains the schema  $\diamond\varphi \leftrightarrow \Box\varphi$  then the canonical model for  $\Sigma$  is functional.*
3. *If  $\Sigma$  contains the schema  $\Box\Box\varphi \rightarrow \Box\varphi$  then the canonical model for  $\Sigma$  is weakly dense.*

(see ?? for definitions of these frame properties).

- Proof.*
1. suppose that  $\Sigma$  contains the schema  $\diamond\varphi \rightarrow \Box\varphi$ , to show that  $R^\Sigma$  is partially functional we need to prove that for any  $\Delta_1, \Delta_2, \Delta_3 \in W^\Sigma$ , if  $R^\Sigma\Delta_1\Delta_2$  and  $R^\Sigma\Delta_1\Delta_3$  then  $\Delta_2 = \Delta_3$ . Since  $R^\Sigma\Delta_1\Delta_2$  we have  $\Box^{-1}\Delta_1 \subseteq \Delta_2$  and since  $R^\Sigma\Delta_1\Delta_3$  also  $\Box^{-1}\Delta_1 \subseteq \Delta_3$ . The identity  $\Delta_2 = \Delta_3$  will follow if we can establish the two inclusions  $\Delta_2 \subseteq \Delta_3$  and  $\Delta_3 \subseteq \Delta_2$ . For the first inclusion, let  $\varphi \in \Delta_2$ ; then  $\diamond\varphi \in \Delta_1$ , and by the schema and deductive closure of  $\Delta_1$  also  $\Box\varphi \in \Delta_1$ , whence by the hypothesis that  $R^\Sigma\Delta_1\Delta_3$ ,  $\varphi \in \Delta_3$ . The second inclusion is similar.
  2. This follows immediately from part (1) and the seriality proof in [Theorem com.1](#).
  3. Suppose  $\Sigma$  contains the schema  $\Box\Box\varphi \rightarrow \Box\varphi$  and to show that  $R^\Sigma$  is weakly dense, let  $R^\Sigma\Delta_1\Delta_2$ . We need to show that there is a complete  $\Sigma$ -consistent set  $\Delta_3$  such that  $R^\Sigma\Delta_1\Delta_3$  and  $R^\Sigma\Delta_3\Delta_2$ . Let:

$$\Gamma = \Box^{-1}\Delta_1 \cup \diamond\Delta_2.$$

It suffices to show that  $\Gamma$  is  $\Sigma$ -consistent, for then by Lindenbaum's Lemma it can be extended to a complete  $\Sigma$ -consistent set  $\Delta_3$  such that  $\Box^{-1}\Delta_1 \subseteq \Delta_3$  and  $\diamond\Delta_2 \subseteq \Delta_3$ , i.e.,  $R^\Sigma\Delta_1\Delta_3$  and  $R^\Sigma\Delta_3\Delta_2$  (by ??).

Suppose for contradiction that  $\Gamma$  is not consistent. Then there are formulas  $\Box\varphi_1, \dots, \Box\varphi_n \in \Delta_1$  and  $\psi_1, \dots, \psi_m \in \Delta_2$  such that

$$\varphi_1, \dots, \varphi_n, \diamond\psi_1, \dots, \diamond\psi_m \vdash_\Sigma \perp.$$

Since  $\Diamond(\psi_1 \wedge \dots \wedge \psi_m) \rightarrow (\Diamond\psi_1 \wedge \dots \wedge \Diamond\psi_m)$  is **derivable** in every normal modal logic, we argue as follows, contradicting the consistency of  $\Delta_2$ :

$$\begin{aligned}
& \varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_{\Sigma} \perp \\
& \Rightarrow \varphi_1, \dots, \varphi_n \vdash_{\Sigma} (\Diamond\psi_1 \wedge \dots \wedge \Diamond\psi_m) \rightarrow \perp, & \text{deduction theorem;} \\
& \Rightarrow \varphi_1, \dots, \varphi_n \vdash_{\Sigma} \Diamond(\psi_1 \wedge \dots \wedge \psi_m) \rightarrow \perp, & \Sigma \text{ is normal;} \\
& \Rightarrow \varphi_1, \dots, \varphi_n \vdash_{\Sigma} \Box\neg(\psi_1 \wedge \dots \wedge \psi_m), & \text{PL;} \\
& \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n \vdash_{\Sigma} \Box\Box\neg(\psi_1 \wedge \dots \wedge \psi_m), & \text{??;} \\
& \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n \vdash_{\Sigma} \Box\neg(\psi_1 \wedge \dots \wedge \psi_m), & \text{by the schema;} \\
& \Rightarrow \Delta_1 \vdash_{\Sigma} \Box\neg(\psi_1 \wedge \dots \wedge \psi_m), & \text{Monotony;} \\
& \Rightarrow \Box\neg(\psi_1 \wedge \dots \wedge \psi_m) \in \Delta_1, & \text{deductive closure;} \\
& \Rightarrow \neg(\psi_1 \wedge \dots \wedge \psi_m) \in \Delta_2, & \text{since } R^{\Sigma} \Delta_1 \Delta_2.
\end{aligned}$$

□

On the strength of these examples, one might think that every system  $\Sigma$  of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of  $\Sigma$  is valid. Unfortunately, there are many systems that are not complete in this sense.

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## Bibliography