Frame Completeness

The completeness theorem for K can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

Theorem com.1. If a normal modal logic $\Sigma$ contains one of the formulas on the left-hand side of table 1, then the canonical model for $\Sigma$ has the corresponding property on the right-hand side.

<table>
<thead>
<tr>
<th>If $\Sigma$ contains ...</th>
<th>... the canonical model for $\Sigma$ is:</th>
</tr>
</thead>
<tbody>
<tr>
<td>D: $\Box \varphi \rightarrow \Diamond \varphi$</td>
<td>serial;</td>
</tr>
<tr>
<td>T: $\Box \varphi \rightarrow \varphi$</td>
<td>reflexive;</td>
</tr>
<tr>
<td>B: $\varphi \rightarrow \Box \Diamond \varphi$</td>
<td>symmetric;</td>
</tr>
<tr>
<td>4: $\Box \varphi \rightarrow \Box \Diamond \varphi$</td>
<td>transitive;</td>
</tr>
<tr>
<td>5: $\Diamond \varphi \rightarrow \Box \Diamond \varphi$</td>
<td>euclidean.</td>
</tr>
</tbody>
</table>

Table 1: Basic correspondence facts.

Proof. We take each of these up in turn.

Suppose $\Sigma$ contains D, and let $\Delta \in W^\Sigma$; we need to show that there is a $\Delta'$ such that $R^\Sigma \Delta \Delta'$. It suffices to show that $\square^{-1} \Delta$ is $\Sigma$-consistent, for then by Lindenbaum’s Lemma, there is a complete $\Sigma$-consistent set $\Delta' \supseteq \square^{-1} \Delta$, and by definition of $R^\Sigma$ we have $R^\Sigma \Delta \Delta'$. So, suppose for contradiction that $\square^{-1} \Delta$ is not $\Sigma$-consistent, i.e., $\square^{-1} \Delta \vdash_{\Sigma} \bot$. By $\square^\perp$, $\Delta \vdash_{\Sigma} \square \bot$, and since $\Sigma$ contains D, also $\Delta \vdash_{\Sigma} \Diamond \bot$. But $\Sigma$ is normal, so $\Sigma \vdash \neg \Diamond \bot$ (??), whence also $\Delta \vdash_{\Sigma} \neg \Diamond \bot$, against the consistency of $\Delta$.

Now suppose $\Sigma$ contains T, and let $\Delta \in W^\Sigma$. We want to show $R^\Sigma \Delta \Delta$, i.e., $\square^{-1} \Delta \subseteq \Delta$. But if $\Box \varphi \in \Delta$ then by T also $\varphi \in \Delta$, as desired.

Now suppose $\Sigma$ contains B, and suppose $R^\Sigma \Delta \Delta'$ for $\Delta, \Delta' \in W^\Sigma$. We need to show that $R^\Sigma \Delta \Delta'$, i.e., $\square^{-1} \Delta \subseteq \Delta$. By $\Diamond \bot$, this is equivalent to $\Diamond \Delta \subseteq \Delta'$. So suppose $\varphi \in \Delta$. By B, also $\Box \Diamond \varphi \in \Delta$. By the hypothesis that $R^\Sigma \Delta \Delta'$, we have that $\square^{-1} \Delta \subseteq \Delta'$, and hence $\Diamond \varphi \in \Delta'$, as required.

Now suppose $\Sigma$ contains 4, and suppose $R^\Sigma \Delta_1 \Delta_2$ and $R^\Sigma \Delta_2 \Delta_3$. We need to show $R^\Sigma \Delta_1 \Delta_3$. From the hypothesis we have both $\square^{-1} \Delta_1 \subseteq \Delta_2$ and $\square^{-1} \Delta_2 \subseteq \Delta_3$. In order to show $R^\Sigma \Delta_1 \Delta_3$ it suffices to show $\square^{-1} \Delta_1 \subseteq \Delta_3$. So let $\psi \in \square^{-1} \Delta_1$, i.e., $\Box \psi \in \Delta_1$. By 4, also $\Box \Box \psi \in \Delta_1$ and by hypothesis we get, first, that $\Box \psi \in \Delta_2$ and, second, that $\psi \in \Delta_3$, as desired.

Now suppose $\Sigma$ contains 5, suppose $R^\Sigma \Delta_1 \Delta_2$ and $R^\Sigma \Delta_2 \Delta_3$. We need to show $R^\Sigma \Delta_1 \Delta_3$. The first hypothesis gives $\square^{-1} \Delta_1 \subseteq \Delta_2$, and the second hypothesis is equivalent to $\Diamond \Delta_3 \subseteq \Delta_2$, by ???. To show $R^\Sigma \Delta_2 \Delta_3$, by ???, it suffices to show $\Diamond \Delta_3 \subseteq \Delta_2$. So let $\varphi \in \Diamond \Delta_3$, i.e., $\varphi \in \Delta_3$. By the second hypothesis $\Diamond \varphi \in \Delta_1$ and by 5, $\Box \Diamond \varphi \in \Delta_1$ as well. But now the first hypothesis gives $\Diamond \varphi \in \Delta_2$, as desired. $\Box$

As a corollary we obtain completeness results for a number of systems. For instance, we know that S5 = KT5 = KTB4 is complete with respect to the
class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

**Theorem com.2.** Let $C_D$, $C_T$, $C_B$, $C_4$, and $C_5$ be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1, \ldots, \varphi_n$ among $D$, $T$, $B$, $4$, and $5$, the system $K\varphi_1 \ldots \varphi_n$ is determined by the class of models $C = C_{\varphi_1} \cap \cdots \cap C_{\varphi_n}$.

**Proposition com.3.** Let $\Sigma$ be a normal modal logic; then:

1. If $\Sigma$ contains the schema $\Diamond \varphi \rightarrow \Box \varphi$ then the canonical model for $\Sigma$ is partially functional.
2. If $\Sigma$ contains the schema $\Diamond \varphi \leftrightarrow \Box \varphi$ then the canonical model for $\Sigma$ is functional.
3. If $\Sigma$ contains the schema $\Box \Box \varphi \rightarrow \Box \varphi$ then the canonical model for $\Sigma$ is weakly dense.

*(see ?? for definitions of these frame properties).*

**Proof.** 1. Suppose that $\Sigma$ contains the schema $\Diamond \varphi \rightarrow \Box \varphi$, to show that $R^K$ is partially functional we need to prove that for any $\Delta_1, \Delta_2, \Delta_3 \in W^K$, if $R^K \Delta_1 \Delta_2$ and $R^K \Delta_1 \Delta_3$ then $\Delta_2 = \Delta_3$. Since $R^K \Delta_1 \Delta_2$ we have $\Box^{-1} \Delta_1 \subseteq \Delta_2$ and since $R^K \Delta_1 \Delta_3$ also $\Box^{-1} \Delta_1 \subseteq \Delta_3$. The identity $\Delta_2 = \Delta_3$ will follow if we can establish the two inclusions $\Delta_2 \subseteq \Delta_3$ and $\Delta_3 \subseteq \Delta_2$. For the first inclusion, let $\varphi \in \Delta_2$; then $\Diamond \varphi \in \Delta_1$, and by the schema and deductive closure of $\Delta_1$ also $\Box \varphi \in \Delta_1$, whence by the hypothesis that $R^K \Delta_1 \Delta_3$, $\varphi \in \Delta_3$. The second inclusion is similar.

2. This follows immediately from part (1) and the seriality proof in Theorem com.1.

3. Suppose $\Sigma$ contains the schema $\Box \Box \varphi \rightarrow \Box \varphi$ and to show that $R^K$ is weakly dense, let $R^K \Delta_1 \Delta_2$. We need to show that there is a complete $\Sigma$-consistent set $\Delta_3$ such that $R^K \Delta_1 \Delta_3$ and $R^K \Delta_3 \Delta_2$. Let:

$$\Gamma = \Box^{-1} \Delta_1 \cup \Diamond \Delta_2.$$

It suffices to show that $\Gamma$ is $\Sigma$-consistent, for then by Lindenbaum’s Lemma it can be extended to a complete $\Sigma$-consistent set $\Delta_3$ such that $\Box^{-1} \Delta_1 \subseteq \Delta_3$ and $\Diamond \Delta_2 \subseteq \Delta_3$, i.e., $R^K \Delta_1 \Delta_3$ and $R^K \Delta_3 \Delta_2$ (by ??).

Suppose for contradiction that $\Gamma$ is not consistent. Then there are formulas $\Box \varphi_1, \ldots, \Box \varphi_n \in \Delta_1$ and $\psi_1, \ldots, \psi_m \in \Delta_2$ such that

$$\varphi_1, \ldots, \varphi_n, \Diamond \psi_1, \ldots, \Diamond \psi_m \not\vdash_{\Sigma} \bot.$$
Since $\Diamond (\psi_1 \land \cdots \land \psi_m) \rightarrow (\Diamond \psi_1 \land \cdots \land \Diamond \psi_m)$ is derivable in every normal modal logic, we argue as follows, contradicting the consistency of $\Delta_2$:

$$\varphi_1, \ldots, \varphi_n, \Diamond \psi_1, \ldots, \Diamond \psi_m \vdash_\Sigma \bot$$
$$\varphi_1, \ldots, \varphi_n \vdash_\Sigma (\Diamond \psi_1 \land \cdots \land \Diamond \psi_m) \rightarrow \bot$$

by the deduction theorem

$\text{TAUT}$, and

$\varphi_1, \ldots, \varphi_n \vdash_\Sigma \Diamond (\psi_1 \land \cdots \land \psi_m) \rightarrow \bot$

since $\Sigma$ is normal

$\varphi_1, \ldots, \varphi_n \vdash_\Sigma \neg \Diamond (\psi_1 \land \cdots \land \psi_m)$

by PL

$\varphi_1, \ldots, \varphi_n \vdash_\Sigma \Box \neg (\psi_1 \land \cdots \land \psi_m)$

$\Box \neg$ for $\neg \Diamond$

$\Box \varphi_1, \ldots, \Box \varphi_n \vdash_\Sigma \Box \Box \neg (\psi_1 \land \cdots \land \psi_m)$

by $\Box$?

$\Box \varphi_1, \ldots, \Box \varphi_n \vdash_\Sigma \Box \neg (\psi_1 \land \cdots \land \psi_m)$

by schema $\Box \Box \varphi \rightarrow \Box \varphi$

$\Delta_1 \vdash_\Sigma \Box \neg (\psi_1 \land \cdots \land \psi_m)$

by monotony, $\text{TAUT}$

$\Box \neg (\psi_1 \land \cdots \land \psi_m) \in \Delta_1$

by deductive closure;

$\neg (\psi_1 \land \cdots \land \psi_m) \in \Delta_2$

since $R^\Sigma \Delta_1 \Delta_2$.

On the strength of these examples, one might think that every system $\Sigma$ of modal logic is complete, in the sense that it proves every formula which is valid in every frame in which every theorem of $\Sigma$ is valid. Unfortunately, there are many systems that are not complete in this sense.

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Bibliography