com.1 Frame Completeness

The completeness theorem for \( \mathbf{K} \) can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

**Theorem com.1.** If a normal modal logic \( \Sigma \) contains one of the formulas on the left-hand side of table 1, then the canonical model for \( \Sigma \) has the corresponding property on the right-hand side.

<table>
<thead>
<tr>
<th>If ( \Sigma ) contains ...</th>
<th>... the canonical model for ( \Sigma ) is:</th>
</tr>
</thead>
<tbody>
<tr>
<td>D: ( \square \phi \to \phi )</td>
<td>serial;</td>
</tr>
<tr>
<td>T: ( \square \phi \to \phi )</td>
<td>reflexive;</td>
</tr>
<tr>
<td>B: ( \phi \to \square \phi )</td>
<td>symmetric;</td>
</tr>
<tr>
<td>4: ( \square \phi \to \square \square \phi )</td>
<td>transitive;</td>
</tr>
<tr>
<td>5: ( \Diamond \phi \to \square \Diamond \phi )</td>
<td>euclidean.</td>
</tr>
</tbody>
</table>

Table 1: Basic correspondence facts.

*Proof.* We take each of these up in turn.

Suppose \( \Sigma \) contains D, and let \( \Delta \in W^\Sigma \); we need to show that there is a \( \Delta' \) such that \( R^\Sigma \Delta \Delta' \). It suffices to show that \( \square^{-1} \Delta \) is \( \Sigma \)-consistent, for then by Lindenbaum’s Lemma, there is a complete \( \Sigma \)-consistent set \( \Delta' \supseteq \square^{-1} \Delta \), and by definition of \( R^\Sigma \) we have \( R^\Sigma \Delta \Delta' \). So, suppose for contradiction that \( \square^{-1} \Delta \) is *not* \( \Sigma \)-consistent, i.e., \( \square^{-1} \Delta \vdash \perp \). By \( \square \), \( \Delta \vdash \perp \), and since \( \Sigma \) contains D, also \( \Delta \vdash \Diamond \perp \). But \( \Sigma \) is normal, so \( \Sigma \vdash \neg \Diamond \perp \), whence also \( \Delta \vdash \neg \Diamond \perp \), against the consistency of \( \Delta \).

Now suppose \( \Sigma \) contains T, and let \( \Delta \in W^\Sigma \). We want to show \( R^\Sigma \Delta \Delta \), i.e., \( \square^{-1} \Delta \subseteq \Delta \). But if \( \square \phi \in \Delta \) then by T also \( \phi \in \Delta \), as desired.

Now suppose \( \Sigma \) contains B, and suppose \( R^\Sigma \Delta \Delta' \) for \( \Delta, \Delta' \in W^\Sigma \). We need to show that \( R^\Sigma \Delta \Delta \), i.e., \( \square^{-1} \Delta \subseteq \Delta \). By \( \square \), this is equivalent to \( \Diamond \Delta \subseteq \Delta' \).

Suppose \( \phi \in \Delta \). By B, also \( \square \Diamond \phi \in \Delta \). By the hypothesis that \( R^\Sigma \Delta \Delta' \), we have that \( \square^{-1} \Delta \subseteq \Delta' \), and hence \( \Diamond \phi \in \Delta' \), as required.

Now suppose \( \Sigma \) contains 4, and suppose \( R^\Sigma \Delta_1 \Delta_2 \) and \( R^\Sigma \Delta_2 \Delta_3 \). We need to show \( R^\Sigma \Delta_1 \Delta_3 \). From the hypothesis we have both \( \square^{-1} \Delta_1 \subseteq \Delta_2 \) and \( \square^{-1} \Delta_2 \subseteq \Delta_3 \). In order to show \( R^\Sigma \Delta_1 \Delta_3 \) it suffices to show \( \square^{-1} \Delta_1 \subseteq \Delta_3 \). So let \( \psi \in \square^{-1} \Delta_1 \), i.e., \( \square \psi \in \Delta_1 \). By 4, also \( \square \square \psi \in \Delta_1 \) and by hypothesis we get, first, that \( \square \psi \in \Delta_2 \) and, second, that \( \psi \in \Delta_3 \), as desired.

Now suppose \( \Sigma \) contains 5, suppose \( R^\Sigma \Delta_1 \Delta_2 \) and \( R^\Sigma \Delta_1 \Delta_3 \). We need to show \( R^\Sigma \Delta_2 \Delta_3 \). The first hypothesis gives \( \square^{-1} \Delta_1 \subseteq \Delta_2 \) and the second hypothesis is equivalent to \( \Diamond \Delta_1 \subseteq \Delta_2 \), by \( \square \). To show \( R^\Sigma \Delta_2 \Delta_3 \), by \( \square \), it suffices to show \( \Diamond \Delta_1 \subseteq \Delta_3 \). So let \( \Diamond \phi \in \Diamond \Delta_3 \), i.e., \( \phi \in \Delta_3 \). By the second hypothesis \( \Diamond \phi \in \Delta_1 \) and by 5, \( \square \Diamond \phi \in \Delta_1 \) as well. But now the first hypothesis gives \( \Diamond \phi \in \Delta_2 \), as desired. \( \square \)

As a corollary we obtain completeness results for a number of systems. For instance, we know that \( \mathbf{S}5 = \mathbf{K} \mathbf{T}5 = \mathbf{K} \mathbf{T}B4 \) is complete with respect to the...
class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

**Theorem com.2.** Let \( C_D, C_T, C_B, C_4, \) and \( C_5 \) be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas \( \varphi_1, \ldots, \varphi_n \) among \( D, T, B, 4, \) and \( 5 \), the system \( K\varphi_1 \ldots \varphi_n \) is determined by the class of models \( C = C_{\varphi_1} \cap \cdots \cap C_{\varphi_n} \).

**Proposition com.3.** Let \( \Sigma \) be a normal modal logic; then:

1. If \( \Sigma \) contains the schema \( \Diamond \varphi \rightarrow \Box \varphi \) then the canonical model for \( \Sigma \) is partially functional.
2. If \( \Sigma \) contains the schema \( \Diamond \varphi \leftrightarrow \Box \varphi \) then the canonical model for \( \Sigma \) is functional.
3. If \( \Sigma \) contains the schema \( \Box \Box \varphi \rightarrow \Box \varphi \) then the canonical model for \( \Sigma \) is weakly dense.

(see ?? for definitions of these frame properties).

**Proof.** 1. suppose that \( \Sigma \) contains the schema \( \Diamond \varphi \rightarrow \Box \varphi \), to show that \( R^\Sigma \) is partially functional we need to prove that for any \( \Delta_1, \Delta_2, \Delta_3 \in W^\Sigma \), if \( R^\Sigma \Delta_1 \Delta_2 \) and \( R^\Sigma \Delta_1 \Delta_3 \) then \( \Delta_2 = \Delta_3 \). Since \( R^\Sigma \Delta_1 \Delta_2 \) we have \( \Box^{-1} \Delta_1 \subseteq \Delta_2 \) and since \( R^\Sigma \Delta_1 \Delta_3 \) also \( \Box^{-1} \Delta_1 \subseteq \Delta_3 \). The identity \( \Delta_2 = \Delta_3 \) will follow if we can establish the two inclusions \( \Delta_2 \subseteq \Delta_3 \) and \( \Delta_3 \subseteq \Delta_2 \). For the first inclusion, let \( \varphi \in \Delta_2 \); then \( \Diamond \varphi \in \Delta_1 \), and by the schema and deductive closure of \( \Delta_1 \) also \( \Box \varphi \in \Delta_1 \), whence by the hypothesis that \( R^\Sigma \Delta_1 \Delta_3 \), \( \varphi \in \Delta_3 \). The second inclusion is similar.

2. This follows immediately from part (1) and the seriality proof in Theorem com.1.

3. Suppose \( \Sigma \) contains the schema \( \Box \Box \varphi \rightarrow \Box \varphi \) and to show that \( R^\Sigma \) is weakly dense, let \( R^\Sigma \Delta_1 \Delta_2 \). We need to show that there is a complete \( \Sigma \)-consistent set \( \Delta_3 \) such that \( R^\Sigma \Delta_1 \Delta_3 \) and \( R^\Sigma \Delta_3 \Delta_2 \). Let:

\[
\Gamma = \Box^{-1} \Delta_1 \cup \Diamond \Delta_2.
\]

It suffices to show that \( \Gamma \) is \( \Sigma \)-consistent, for then by Lindenbaum’s Lemma it can be extended to a complete \( \Sigma \)-consistent set \( \Delta_3 \) such that \( \Box^{-1} \Delta_1 \subseteq \Delta_3 \) and \( \Diamond \Delta_2 \subseteq \Delta_3 \), i.e., \( R^\Sigma \Delta_1 \Delta_3 \) and \( R^\Sigma \Delta_3 \Delta_2 \) (by ??).

Suppose for contradiction that \( \Gamma \) is not consistent. Then there are formulas \( \Box \varphi_1, \ldots, \Box \varphi_n \in \Delta_1 \) and \( \psi_1, \ldots, \psi_m \in \Delta_2 \) such that

\[
\varphi_1, \ldots, \varphi_n, \Diamond \psi_1, \ldots, \Diamond \psi_m \vdash_\Sigma \bot.
\]
Since $\Diamond (\psi_1 \land \cdots \land \psi_m) \rightarrow (\Diamond \psi_1 \land \cdots \land \Diamond \psi_m)$ is derivable in every normal modal logic, we argue as follows, contradicting the consistency of $\Delta_2$:

$$\varphi_1, \ldots, \varphi_n, \Diamond \psi_1, \ldots, \Diamond \psi_m \vdash \Sigma \bot \Rightarrow \varphi_1, \ldots, \varphi_n \vdash (\Diamond \psi_1 \land \cdots \land \Diamond \psi_m) \rightarrow \bot,$$

deduction theorem;

$$\Rightarrow \varphi_1, \ldots, \varphi_n \vdash \Sigma (\Diamond \psi_1 \land \cdots \land \psi_m) \rightarrow \bot,$$

$\Sigma$ is normal;

$$\Rightarrow \varphi_1, \ldots, \varphi_n \vdash \Sigma \Box \neg (\psi_1 \land \cdots \land \psi_m),$$

PL;

$$\Rightarrow \Box \varphi_1, \ldots, \Box \varphi_n \vdash \Sigma \Box \neg (\psi_1 \land \cdots \land \psi_m),$$

by the schema;

$$\Rightarrow \Delta_1 \vdash \Sigma \Box \neg (\psi_1 \land \cdots \land \psi_m),$$

Monotony;

$$\Rightarrow \Box \neg (\psi_1 \land \cdots \land \psi_m) \in \Delta_1,$$

deductive closure;

$$\Rightarrow \neg (\psi_1 \land \cdots \land \psi_m) \in \Delta_2,$$

since $R^\Sigma \Delta_1 \Delta_2$.

\[\Box\]

On the strength of these examples, one might think that every system $\Sigma$ of modal logic is complete, in the sense that it proves every formula which is valid in every frame in which every theorem of $\Sigma$ is valid. Unfortunately, there are many systems that are not complete in this sense.

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Bibliography