Chapter udf

Completeness

com.1 Complete Consistent Sets

mod:com:ccs:sec

Definition com.1. A set Γ is *complete Σ-consistent* if and only if it is Σ-consistent and for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

mod:com:ccs:prop:completeconsproperties

Proposition com.2. *Suppose Γ is complete Σ-consistent. Then:*

1. *Γ is deductively closed in Σ.*
2. *$\Sigma \subseteq \Gamma$.* (mod:com:ccs:prop:completeconsproperties-b)
3. *$\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.* (mod:com:ccs:prop:completeconsproperties-c)
4. *$\varphi \to \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ implies $\psi \in \Gamma$.* (mod:com:ccs:prop:completeconsproperties-d)

Proof.

1. If $\Gamma \vdash_\Sigma \varphi$ but $\varphi \notin \Gamma$ then by maximality $\neg\varphi \in \Gamma$, and Γ is inconsistent.
2. If $\varphi \in \Sigma$ then $\Gamma \vdash_\Sigma \varphi$, and $\varphi \in \Gamma$ by deductive closure.
3. If $\neg\varphi \in \Gamma$, then by consistency $\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then by maximality $\varphi \in \Gamma$.
4. Suppose $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash_\Sigma \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \to \psi \notin \Gamma$ then by maximality $\neg(\varphi \to \psi) \in \Gamma$, so by Rule T, deductive closure, and consistency both $\varphi \in \Gamma$ and $\psi \notin \Gamma$.

□

com.2 Lindenbaum's Lemma

mod:com:lin:sec

Theorem com.3 (Lindenbaum's Lemma)**.** *If Γ is Σ-consistent then there is a complete Σ-consistent set Δ extending Γ.* mod:com:lin:thm:lindenbaum

Proof. Let φ_0, φ_1, ... be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing p_0, and at each stage n list the finitely many formulas of length n using only variables among p_0, ..., p_n. We define sets of formulas Δ_n by induction on n, and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$, then supposing that Δ_n has been defined:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

If we now let $\Delta = \bigcup_n \Delta_n$, we can show the following:

1. For each n, $\Delta_n \subseteq \Delta$ (immediate from the definition).
2. $\Gamma \subseteq \Delta$ (from (a)).
3. If $n \leq m$ then $\Delta_n \subseteq \Delta_m$ (by induction on $m - n$).
4. Δ is maximal (by construction).
5. For each m, Δ_m is consistent (by induction on m, using **????**)
6. If $\Delta' \subseteq \Delta$ is finite, then there is m such that $\Delta' \subseteq \Delta_m$.
7. Δ is consistent.

It follows that Δ is a complete Σ-consistent set extending Γ. □

com.3 Derivability and Complete Consistent Sets

mod:com:pcc:sec

Corollary com.4. *$\Gamma \vdash_\Sigma \varphi$ if and only if $\varphi \in \Delta$ for each complete Σ-consistent set Δ extending Γ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system Σ.)* mod:com:pcc:cor:provability-characterization

Proof. Suppose $\Gamma \vdash_\Sigma \varphi$, and let Δ be any complete Σ-consistent set extending Γ. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_\Sigma \varphi$ (by monotony) and $\Delta \vdash_\Sigma \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \nvdash_\Sigma \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent, and by Lindenbaum's Lemma there is a complete consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$. By consistency, $\varphi \notin \Delta$. □

com.4 Modalities and Complete Consistent Sets

mod:com:mod:sec When we construct a model whose set of worlds is given by the complete consistent sets in some normal modal logic Σ, we will also need to define an accessibility relation between such "worlds." The next few lemmas give us the tools to do so. As noted, Σ will be a normal modal logic throughout.

mod:com:mod:lem:Gamma-proves1 **Lemma com.5.** *If $\Gamma \vdash_\Sigma \varphi$ then $\{\Box\psi : \psi \in \Gamma\} \vdash_\Sigma \Box\varphi$.*

Proof. If $\Gamma \vdash_\Sigma \varphi$ then there are $\psi_1, \ldots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots(\psi_n \to \varphi)\cdots)$. Since Σ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \to (\Box\psi_2 \to \cdots(\Box\psi_n \to \Box\varphi)\cdots)$, where obviously $\Box\psi_1, \ldots, \Box\psi_k \in \{\Box\psi : \psi \in \Gamma\}$. Hence, by definition, $\{\Box\psi : \psi \in \Gamma\} \vdash_\Sigma \Box\varphi$. □

mod:com:mod:lem:Gamma-proves2 **Lemma com.6.** *If $\{\psi : \Box\psi \in \Gamma\} \vdash_\Sigma \varphi$ then $\Gamma \vdash_\Sigma \Box\varphi$.*

Proof. Let $\Delta = \{\psi : \Box\psi \in \Gamma\}$, so that $\Delta \vdash_\Sigma \varphi$; then by Lemma com.5, $\{\Box\psi : \psi \in \Delta\} \vdash \Box\varphi$. But obviously $\{\Box\psi : \psi \in \Delta\} \subseteq \Gamma$, so also $\Gamma \vdash_\Sigma \Box\varphi$ by Monotony. □

mod:com:mod:thm:box-phiGamma **Theorem com.7.** *If Γ is complete Σ-consistent, then $\Box\varphi \in \Gamma$ if and only if for every complete Σ-consistent Δ such that $\{\psi : \Box\psi \in \Gamma\} \subseteq \Delta$, it holds that $\varphi \in \Delta$.*

Proof. The left-to-right half of the theorem is obvious. For the converse, suppose $\Box\varphi \notin \Gamma$. Since Γ is deductively closed, $\Gamma \nvdash_\Sigma \Box\varphi$, and by Lemma com.6 $\{\psi : \Box\psi \in \Gamma\} \nvdash_\Sigma \varphi$. By **????**, $\{\psi : \Box\psi \in \Gamma\} \cup \{\neg\varphi\}$ is Σ-consistent, so that by Lindenbaum's Lemma there is a complete Σ-consistent set Δ such that $\{\psi : \Box\psi \in \Gamma\} \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$, and the theorem is proved. □

mod:com:mod:lem:Gamma-proves3 **Lemma com.8.** *Suppose Γ and Δ are complete Σ-consistent. Then: $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$ if and only if $\{\Diamond\varphi : \varphi \in \Delta\} \subseteq \Gamma$.*

Proof. "Only if" direction: Assume $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$ and suppose $\varphi \in \Delta$. In order to show $\Diamond\varphi \in \Gamma$ it suffices to show $\Box\neg\varphi \notin \Gamma$ for then by maximality $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of Δ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

"If" direction: Assume $\{\Diamond\varphi : \varphi \in \Delta\} \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in Γ by consistency $\Box\varphi \notin \Gamma$, as required. □

Corollary com.9. *If Γ is complete Σ-consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some complete Σ-consistent Δ such that $\{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma$, it holds that $\varphi \in \Delta$.*

Proof. Suppose Γ is complete Σ-consistent, and argue as follows:

$$\begin{aligned}
\Diamond\varphi \in \Gamma \quad &\Leftrightarrow \quad \neg\Box\neg\varphi \in \Gamma, && \text{re-writing;}\\
&\Leftrightarrow \quad \Box\neg\varphi \notin \Gamma, && \Gamma \text{ is complete } \Sigma\text{-co}\\
&\Leftrightarrow \quad \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\psi : \Box\psi \in \Gamma\} \subseteq \Delta \wedge \neg\varphi \notin \Delta]\,, && \text{Theorem com.7;}\\
&\Leftrightarrow \quad \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma \wedge \neg\varphi \notin \Delta]\,, && \text{Lemma com.8;}\\
&\Leftrightarrow \quad \exists\Delta\,[\Delta \text{ is complete } \Sigma\text{-consistent} \wedge \{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma \wedge \varphi \in \Delta]\,, \Box && \Delta \text{ is complete.}
\end{aligned}$$

com.5 Canonical Models

mod:com:cmd: sec

Definition com.10. A model $\mathfrak{M}$ is said to *determine* a normal modal logic Σ precisely when $\mathfrak{M} \models \varphi$ if and only if $\Sigma \vdash \varphi$, for all formulas φ.

Definition com.11. Let Σ be a normal modal logic. The *canonical model* for Σ is $\mathfrak{M}^\Sigma = \langle W^\Sigma, R^\Sigma, V^\Sigma\rangle$, where:

1. $\mathfrak{M}^\Sigma = \{w \subseteq \mathrm{Frm}(\mathcal{L}) : w \text{ is complete } \Sigma\text{-consistent}\}$.
2. $R^\Sigma ww'$ holds if and only if $\{\varphi : \Box\varphi \in w\} \subseteq w'$.
3. $V^\Sigma(p) = \{w : p \in w\}$.

com.6 The Truth Lemma

mod:com:tru: sec

Proposition com.12 (Truth Lemma)**.** *For every formula φ, $\mathfrak{M}^\Sigma, w \models \varphi$ if and only if $\varphi \in w$.* *mod:com:tru: prop:truthlemma*

Proof. By induction on φ. *Basis*: if φ is a propositional variable, say p, then:

$$\mathfrak{M}^\Sigma, w \models p \Leftrightarrow w \in V^\Sigma(p) \Leftrightarrow p \in w.$$

If φ is $\bot$ then both $\mathfrak{M}^\Sigma, w \not\models \bot$ and $\bot \notin w$ (by consistency of w). The cases for $\neg\varphi$ and $\varphi \to \psi$ follow from the inductive hypothesis and Proposition com.2, parts Proposition com.2 and Proposition com.2. Here is the case for $\Box\varphi$; in one direction:

$$\begin{aligned}
\mathfrak{M}^\Sigma, w \models \Box\varphi \quad &\Rightarrow \quad \forall w' \in W^\Sigma(R^\Sigma ww' \Rightarrow \mathfrak{M}^\Sigma, w' \models \varphi), && \text{def. } \models;\\
&\Rightarrow \quad \forall w' \in W^\Sigma(\{\psi : \Box\psi \in w\} \subseteq w' \Rightarrow \mathfrak{M}^\Sigma, w' \models \varphi), && \text{def. } R^\Sigma;\\
&\Rightarrow \quad \forall w' \in W^\Sigma(\{\psi : \Box\psi \in w\} \subseteq w' \Rightarrow \varphi \in w'), && \text{ind. hyp.;}\\
&\Rightarrow \quad \Box\varphi \in w, && \text{Theorem com.7.}
\end{aligned}$$

Conversely, assume $\Box\varphi \in w$, and let w' be an arbitrary world in W^Σ such that $R^\Sigma ww'$. By definition of R^Σ, we have $\{\psi : \Box\psi \in w\} \subseteq w'$, which immediately gives $\varphi \in w'$. By induction hypothesis, $\mathfrak{M}^\Sigma, w' \models \varphi$, and since w' was arbitrary, $\mathfrak{M}^\Sigma, w \models \Box\varphi$. □

com.7 Completeness for K

mod:com:cmk:
sec

mod:com:cmk:
thm:determination

Theorem com.13 (Determination)**.** *For every normal modal logic Σ: $\mathfrak{M}^{\Sigma} \models \varphi$ if and only if $\Sigma \vdash \varphi$.*

Proof. If $\mathfrak{M}^{\Sigma} \models \varphi$, then for every complete Σ-consistent w, we have $\mathfrak{M}^{\Sigma}, w \models \varphi$. Hence, by the Truth Lemma, $\varphi \in w$ for every complete Σ-consistent w, whence by Corollary com.4 (with $\Gamma = \emptyset$), $\Sigma \vdash \varphi$. Conversely, if $\Sigma \vdash \varphi$ then by Proposition com.2(2), every complete Σ-consistent w contains φ, and hence by the Truth Lemma $\mathfrak{M}^{\Sigma}, w \models \varphi$ for every w, i.e., $\mathfrak{M}^{\Sigma} \models \varphi$. $\square$

mod:com:cmk:
cor:Kcomplete

Corollary com.14. *The basic modal logic* **K** *is complete with respect to the class of all models, i.e., if $\models \varphi$ then* $\mathbf{K} \vdash \varphi$.

Proof. Contrapositively, if $\mathbf{K} \nvdash \varphi$ then by Determination $\mathfrak{M}^{\mathbf{K}} \not\models \varphi$ and hence φ is not valid. $\square$

com.8 Frame Completeness

mod:com:fra:
sec

The completeness theorem for LogK can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

mod:com:fra:
thm:completeframeprops

Theorem com.15. *If a normal modal logic Σ contains one of the schemas on the left-hand side of the table of Figure com.1, then the canonical model for Σ has the corresponding property on the right-hand side.*

If Σ contains ...	*... the canonical model for Σ is:*
D: $\Box\varphi \to \Diamond\varphi$	serial;
T: $\Box\varphi \to \varphi$	reflexive;
B: $\varphi \to \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \to \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \to \Box\Diamond\varphi$	euclidean.

Figure com.1: Basic correspondence facts.

mod:com:fra:
fig:correspondencetable

Proof. We take each of these up in turn. Suppose Σ contains D, and let $w \in W^{\Sigma}$; we need to show that there is a w' such that $R^{\Sigma}ww'$. It suffices to show that $\{\psi : \Box\psi \in w\}$ is Σ-consistent, for then by Lindenbaum's Lemma, there is a complete Σ-consistent set $w' \supseteq \{\psi : \Box\psi \in w\}$, and by definition of R^{Σ} we have $R^{\Sigma}ww'$. So, suppose for contradiction that $\{\psi : \Box\psi \in w\}$ is *not* Σ-consistent, i.e., $\{\psi : \Box\psi \in w\} \vdash_{\Sigma} \bot$. By Lemma com.6, $w \vdash_{\Sigma} \Box\bot$, and since Σ contains D, also $w \vdash_{\Sigma} \Diamond\bot$. But Σ is normal, so $\Sigma \vdash \neg\Diamond\bot$ (**??**), whence also $w \vdash_{\Sigma} \neg\Diamond\bot$, against the consistency of w.

Now suppose Σ contains T, and let $w \in W^{\Sigma}$. We want to show $R^{\Sigma}ww$, i.e., $\{\varphi : \Box\varphi \in w\} \subseteq w$. But if $\Box\varphi \in w$ then by T also $\varphi \in w$, as desired.

Now suppose Σ contains B, and suppose $R^{\Sigma}uv$ for $u, v \in W^{\Sigma}$. We need to show that $R^{\Sigma}vu$, i.e., $\{\varphi : \Box\varphi \in v\} \subseteq u$. By Lemma com.8, this is equivalent to $\{\Diamond\varphi : \varphi \in u\} \subseteq v$. So suppose $\varphi \in u$. By B, also $\Box\Diamond\varphi \in u$. By the hypothesis that $R^{\Sigma}uv$, we have that $\{\psi : \Box\psi \in u\} \subseteq v$, and hence $\Diamond\varphi \in v$, as required.

Now suppose Σ contains 4, and suppose $R^{\Sigma}uv$ and $R^{\Sigma}vw$. We need to show $R^{\Sigma}uw$. From the hypothesis we have both $\{\psi : \Box\psi \in u\} \subseteq v$ and $\{\psi : \Box\psi \in v\} \subseteq w$. In order to show $R^{\Sigma}uw$ it suffices to show $\{\psi : \Box\psi \in u\} \subseteq w$. So let $\Box\psi \in u$; by 4, also $\Box\Box\psi \in u$ and by hypothesis we get, first, that $\Box\psi \in v$ and, second, that $\psi \in w$, as desired.

Now suppose Σ contains 5, suppose $R^{\Sigma}uv$ and $R^{\Sigma}uw$. We need to show $R^{\Sigma}vw$. The first hypothesis give $\{\varphi : \Box\varphi \in u\} \subseteq v$, and the second hypothesis is equivalent to $\{\Diamond\varphi : \varphi \in w\} \subseteq u$, by Lemma com.8. To show $R^{\Sigma}vw$, by Lemma com.8, it suffices to show $\{\Diamond\varphi : \varphi \in w\} \subseteq v$. So let $\varphi \in w$; by the second hypothesis $\Diamond\varphi \in u$ and by 5, $\Box\Diamond\varphi \in u$ as well. But now the first hypothesis give $\Diamond\varphi \in v$, as desired. $\Box$

As a corollary we obtain completeness results for a number of systems. For instance, we know that $\mathbf{S5} = \mathbf{KT5} = \mathbf{KTB4}$ is complete with respect to the class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

Theorem com.16. *Let $\mathcal{C}_{\mathrm{D}}$, $\mathcal{C}_{\mathrm{T}}$, $\mathcal{C}_{\mathrm{B}}$, $\mathcal{C}_4$, and $\mathcal{C}_5$ be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1, \ldots, \varphi_n$ among* D, T, B, 4, *and* 5, *the system $\mathbf{K}\varphi_1 \ldots \varphi_n$ is determined by the class of models $\mathcal{C} = \mathcal{C}_{\varphi_1} \cap \cdots \cap \mathcal{C}_{\varphi_n}$.* *mod:com:fra: thm:generaldet*

Proposition com.17. *Let Σ be a normal modal logic; then:*

1. *If Σ contains the schema $\Diamond\varphi \to \Box\varphi$ then the canonical model for Σ is partially functional.* *mod:com:fra: prop:anotherfive-a*

2. *If Σ contains the schema $\Diamond\varphi \leftrightarrow \Box\varphi$ then the canonical model for Σ is functional.*

3. *If Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ then the canonical model for Σ is weakly dense.*

(see **??** *for definitions of these frame properties).*

Proof. 1. suppose that Σ contains the schema $\Diamond\varphi \to \Box\varphi$, to show that R^{Σ} is partialy functional we need to prove that for any $u, v, w \in W^{\Sigma}$, if $R^{\Sigma}wu$ and $R^{\Sigma}wv$ then $u = v$. Since $R^{\Sigma}wu$ we have $\{\varphi : \Box\varphi \in w\} \subseteq u$ and since $R^{\Sigma}wv$ also $\{\varphi : \Box\varphi \in w\} \subseteq v$. The identity $u = v$ will follow if we can establish the two inclusions $u \subseteq v$ and $v \subseteq u$. For the first inclusion, let $\varphi \in u$; then $\Diamond\varphi \in w$, and by the schema and deductive closure of w also $\Box\varphi \in w$, whence by the hypothesis that $R^{\Sigma}wv$, $\varphi \in v$. The second inclusion is similar, so this establishes part (a).

2. This follows immediately from part (1) and the seriality proof in Theorem com.15.

3. Suppose Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ and to show that R^Σ is weakly dense, let $R^\Sigma uv$. We need to show that there is a complete Σ-consistent set w such that $R^\Sigma uw$ and $R^\Sigma wv$. Let:

$$\Gamma = \{\varphi : \Box\varphi \in u\} \cup \{\Diamond\psi : \psi \in v\}.$$

It suffices to show that Γ is Σ-consistent, for then by Lindenbaum's Lemma it can be extendend to a complete Σ-consistent set w such that $\{\varphi : \Box\varphi \in u\} \subseteq w$ and $\{\Diamond\psi : \psi \in v\} \subseteq w$, i.e., $R^\Sigma uw$ and $R^\Sigma wv$.

Suppose for contradiction that Γ is not consistent. Then there are formulas $\Box\varphi_1, \dots, \Box\varphi_n \in u$ and $\psi_1, \dots, \psi_m \in v$ such that $\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_\Sigma \bot$. Since $\Diamond(\psi_1 \land \dots \land \psi_m) \to (\Diamond\psi_1 \land \dots \land \Diamond\psi_m)$ is derivable in every normal modal logic, we argue as follows, contradicting the consistency of v:

$$\begin{array}{llll}
\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_\Sigma \bot & \Rightarrow & \varphi_1, \dots, \varphi_n \vdash_\Sigma (\Diamond\psi_1 \land \dots \land \Diamond\psi_m) \to \bot, & \text{deduction theorem;} \\
& \Rightarrow & \varphi_1, \dots, \varphi_n \vdash_\Sigma \Diamond(\psi_1 \land \dots \land \psi_m) \to \bot, & \Sigma \text{ is normal;} \\
& \Rightarrow & \varphi_1, \dots, \varphi_n \vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m), & \text{PL, re-writing;} \\
& \Rightarrow & \Box\varphi_1, \dots, \Box\varphi_n \vdash_\Sigma \Box\Box\neg(\psi_1 \land \dots \land \psi_m), & \text{Lemma com.5;} \\
& \Rightarrow & \Box\varphi_1, \dots, \Box\varphi_n \vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m), & \text{by the schema;} \\
& \Rightarrow & u \vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m), & \text{Monotony;} \\
& \Rightarrow & \Box\neg(\psi_1 \land \dots \land \psi_m) \in u, & \text{deductive closure;} \\
& \Rightarrow & \neg(\psi_1 \land \dots \land \psi_m) \in v, \quad \Box & \text{since } R^\Sigma uv.
\end{array}$$

On the strength of these examples, one might think that every system Σ of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of Σ is valid. Unfortunately, there are many systems that are not complete in this sense.

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Bibliography