

Chapter udf

Completeness and Canonical Models

com.1 Introduction

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If Σ is a modal system, then the soundness theorem establishes that if $\Sigma \vdash \varphi$, then φ is valid in any class \mathcal{C} of models in which all instances of all **formulas** in Σ are valid. In particular that means that if $\mathbf{K} \vdash \varphi$ then φ is true in all models; if $\mathbf{KT} \vdash \varphi$ then φ is true in all reflexive models; if $\mathbf{KD} \vdash \varphi$ then φ is true in all serial models, etc.

Completeness is the converse of soundness: that \mathbf{K} is complete means that if a **formula** φ is valid, $\vdash \varphi$, for instance. Proving completeness is a lot harder to do than proving soundness. It is useful, first, to consider the contrapositive: \mathbf{K} is complete iff whenever $\not\vdash \varphi$, there is a countermodel, i.e., a model \mathfrak{M} such that $\mathfrak{M} \not\models \varphi$. Equivalently (negating φ), we could prove that whenever $\not\vdash \neg\varphi$, there is a model of φ . In the construction of such a model, we can use information contained in φ . When we find models for specific **formulas** we often do the same: E.g., if we want to find a countermodel to $p \rightarrow \Box q$, we know that it has to contain a world where p is true and $\Box q$ is false. And a world where $\Box q$ is false means there has to be a world accessible from it where q is false. And that's all we need to know: which worlds make the **propositional variables** true, and which worlds are accessible from which worlds.

In the case of proving completeness, however, we don't have a specific **formula** φ for which we are constructing a model. We want to establish that a model exists for every φ such that $\not\vdash_{\Sigma} \neg\varphi$. This is a minimal requirement, since if $\vdash_{\Sigma} \neg\varphi$, by soundness, there is no model for φ (in which Σ is true). Now note that $\not\vdash_{\Sigma} \neg\varphi$ iff φ is Σ -consistent. (Recall that $\Sigma \not\vdash_{\Sigma} \neg\varphi$ and $\varphi \not\vdash_{\Sigma} \perp$ are equivalent.) So our task is to construct a model for every Σ -consistent **formula**.

The trick we'll use is to find a Σ -consistent set of **formulas** that contains φ , but also other formulas which tell us what the world that makes φ true has to look like. Such sets are *complete* Σ -consistent sets. It's not enough to construct a model with a single world to make φ true, it will have to contain

multiple worlds and an accessibility relation. The complete Σ -consistent set containing φ will also contain other **formulas** of the form $\Box\psi$ and $\Diamond\chi$. In all accessible worlds, ψ has to be true; in at least one, χ has to be true. In order to accomplish this, we'll simply take *all* possible complete Σ -consistent sets as the basis for the set of worlds. A tricky part will be to figure out when a complete Σ -consistent set should count as being accessible from another in our model.

We'll show that in the model so defined, φ is true at a world—which is also a complete Σ -consistent set—iff φ is **an element** of that set. If φ is Σ -consistent, it will be **an element** of at least one complete Σ -consistent set (a fact we'll prove), and so there will be a world where φ is true. So we will have a single model where every Σ -consistent **formula** φ is true at some world. This single model is the *canonical* model for Σ .

com.2 Complete Σ -Consistent Sets

Suppose Σ is a set of modal **formulas**—think of them as the axioms or defining principles of a normal modal logic. A set Γ is Σ -consistent iff $\Gamma \not\vdash_{\Sigma} \perp$, i.e., if there is no **derivation** of $\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_n \rightarrow \perp) \dots)$ from Σ , where each $\varphi_i \in \Gamma$. We will construct a “canonical” model in which each world is taken to be a special kind of Σ -consistent set: one which is not just Σ -consistent, but maximally so, in the sense that it settles the truth value of every modal **formula**: for every φ , either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$:

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Definition com.1. A set Γ is *complete Σ -consistent* if and only if it is Σ -consistent and for every φ , either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Complete Σ -consistent sets Γ have a number of useful properties. For one, they are deductively closed, i.e., if $\Gamma \vdash_{\Sigma} \varphi$ then $\varphi \in \Gamma$. This means in particular that every instance of a **formula** $\varphi \in \Sigma$ is also $\in \Gamma$. Moreover, membership in Γ mirrors the truth conditions for the propositional connectives. This will be important when we define the “canonical model.”

Proposition com.2. *Suppose Γ is complete Σ -consistent. Then:*

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1. Γ is deductively closed in Σ .

[mod:com:ccs:prop:ccs-closed](#)

2. $\Sigma \subseteq \Gamma$.

[mod:com:ccs:prop:ccs-sigma](#)

3. $\perp \notin \Gamma$

[mod:com:ccs:prop:ccs-lfalse](#)

4. $\top \in \Gamma$

[mod:com:ccs:prop:ccs-ltrue](#)

5. $\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.

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6. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$

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7. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$

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8. $\varphi \rightarrow \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$
9. $\varphi \leftrightarrow \psi \in \Gamma$ iff either $\varphi \in \Gamma$ and $\psi \in \Gamma$, or $\varphi \notin \Gamma$ and $\psi \notin \Gamma$

Proof. 1. Suppose $\Gamma \vdash_{\Sigma} \varphi$ but $\varphi \notin \Gamma$. Then since Γ is complete Σ -consistent, $\neg\varphi \in \Gamma$. This would make Γ inconsistent, since $\varphi, \neg\varphi \vdash_{\Sigma} \perp$.

2. If $\varphi \in \Sigma$ then $\Gamma \vdash_{\Sigma} \varphi$, and $\varphi \in \Gamma$ by deductive closure, i.e., case (1).
3. If $\perp \in \Gamma$, then $\Gamma \vdash_{\Sigma} \perp$, so Γ would be Σ -inconsistent.
4. $\Gamma \vdash_{\Sigma} \top$, so $\top \in \Gamma$ by deductive closure, i.e., case (1).
5. If $\neg\varphi \in \Gamma$, then by consistency $\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then $\varphi \in \Gamma$ since Γ is complete Σ -consistent.
6. Suppose $\varphi \wedge \psi \in \Gamma$. Since $(\varphi \wedge \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure, i.e., case (1). Similarly for $\psi \in \Gamma$. On the other hand, suppose both $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then deductive closure implies $(\varphi \wedge \psi) \in \Gamma$, since $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$ is a tautological instance.
7. Suppose $\varphi \vee \psi \in \Gamma$, and $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is complete Σ -consistent, $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. Then $\neg(\varphi \vee \psi) \in \Gamma$ since $\neg\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \vee \psi))$ is a tautological instance. This would mean that Γ is Σ -inconsistent, a contradiction.
8. Suppose $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash_{\Sigma} \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \rightarrow \psi \notin \Gamma$ then since Γ is complete Σ -consistent, $\neg(\varphi \rightarrow \psi) \in \Gamma$. Since $\neg(\varphi \rightarrow \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure. Since $\neg(\varphi \rightarrow \psi) \rightarrow \neg\psi$ is a tautological instance, $\neg\psi \in \Gamma$. Then $\psi \notin \Gamma$ since Γ is Σ -consistent.
9. Suppose $\varphi \leftrightarrow \psi \in \Gamma$. If $\varphi \in \Gamma$, then $\psi \in \Gamma$, since $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$ is a tautological instance. Similarly, if $\psi \in \Gamma$, then $\varphi \in \Gamma$. So either both $\varphi \in \Gamma$ and $\psi \in \Gamma$, or neither $\varphi \in \Gamma$ nor $\psi \in \Gamma$.
Conversely, suppose $\varphi \leftrightarrow \psi \notin \Gamma$. Since Γ is complete Σ -consistent, $\neg(\varphi \leftrightarrow \psi) \in \Gamma$. Since $\neg(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \neg\psi)$ is a tautological instance, if $\varphi \in \Gamma$ then $\neg\psi \in \Gamma$, and since Γ is Σ -consistent, $\psi \notin \Gamma$. Similarly, if $\psi \in \Gamma$ then $\varphi \notin \Gamma$. So neither $\varphi \in \Gamma$ and $\psi \in \Gamma$, nor $\varphi \notin \Gamma$ and $\psi \notin \Gamma$.

□

Problem com.1. Complete the proof of [Proposition com.2](#).

com.3 Lindenbaum's Lemma

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Lindenbaum's Lemma establishes that every Σ -consistent set of formulas is contained in at least one *complete* Σ -consistent set. Our construction of the canonical model will show that for each complete Σ -consistent set Δ , there is a world in the canonical model where all and only the formulas in Δ are true. So

Lindenbaum's Lemma guarantees that every Σ -consistent set is true at some world in the canonical model.

Theorem com.3 (Lindenbaum's Lemma). *If Γ is Σ -consistent then there is a complete Σ -consistent set Δ extending Γ .*

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[thm:lindenbaum](#)

Proof. Let $\varphi_0, \varphi_1, \dots$ be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing ρ_0 , and at each stage $n \geq 1$ list the finitely many formulas of length n using only variables among ρ_0, \dots, ρ_n . We define sets of formulas Δ_n by induction on n , and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$. Supposing that Δ_n has been defined, we define Δ_{n+1} by:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

If we now let $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$.

We have to show that this definition actually yields a set Δ with the required properties, i.e., $\Gamma \subseteq \Delta$ and Δ is complete Σ -consistent.

It's obvious that $\Gamma \subseteq \Delta$, since $\Delta_0 \subseteq \Delta$ by construction, and $\Delta_0 = \Gamma$. In fact, $\Delta_n \subseteq \Delta$ for all n , since Δ is the union of all Δ_n . (Since in each step of the construction, we add a formula to the set already constructed, $\Delta_n \subseteq \Delta_{n+1}$, so since \subseteq is transitive, $\Delta_n \subseteq \Delta_m$ whenever $n \leq m$.) At each stage of the construction, we either add φ_n or $\neg\varphi_n$, and every formula appears (at least once) in the list of all φ_n . So, for every φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$, so Δ is complete by definition.

Finally, we have to show, that Δ is Σ -consistent. To do this, we show that (a) if Δ were Σ -inconsistent, then some Δ_n would be Σ -inconsistent, and (b) all Δ_n are Σ -consistent.

So suppose Δ were Σ -inconsistent. Then $\Delta \vdash_{\Sigma} \perp$, i.e., there are $\varphi_1, \dots, \varphi_k \in \Delta$ such that $\Sigma \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_k \rightarrow \perp) \dots)$. Since $\Delta = \bigcap_{n=0}^{\infty} \Delta_n$, each $\varphi_i \in \Delta_{n_i}$ for some n_i . Let n be the largest of these. Since $n_i \leq n$, $\Delta_{n_i} \subseteq \Delta_n$. So, all φ_i are in some Δ_n . This would mean $\Delta_n \vdash_{\Sigma} \perp$, i.e., Δ_n is Σ -inconsistent.

To show that each Δ_n is Σ -consistent, we use a simple induction on n . $\Delta_0 = \Gamma$, and we assumed Γ was Σ -consistent. So the claim holds for $n = 0$. Now suppose it holds for n , i.e., Δ_n is Σ -consistent. Δ_{n+1} is either $\Delta_n \cup \{\varphi_n\}$ or $\Delta_n \cup \{\neg\varphi_n\}$. In the first case, Δ_{n+1} is clearly Σ -consistent. However, by ????, either $\Delta_n \cup \{\varphi_n\}$ or $\Delta_n \cup \{\neg\varphi_n\}$ is consistent, so Δ_{n+1} is consistent in the other case as well. \square

Corollary com.4. *$\Gamma \vdash_{\Sigma} \varphi$ if and only if $\varphi \in \Delta$ for each complete Σ -consistent set Δ extending Γ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system Σ .)*

[mod:com:lin:](#)
[cor:provability-characterization](#)

Proof. Suppose $\Gamma \vdash_{\Sigma} \varphi$, and let Δ be any complete Σ -consistent set extending Γ . If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_{\Sigma} \varphi$ (by monotony) and

$\Delta \vdash_{\Sigma} \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \not\vdash_{\Sigma} \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ -consistent, and by Lindenbaum's Lemma there is a complete consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$. By consistency, $\varphi \notin \Delta$. \square

com.4 Modalities and Complete Consistent Sets

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When we construct a model \mathfrak{M}^{Σ} whose set of worlds is given by the complete Σ -consistent sets Δ in some normal modal logic Σ , we will also need to define an accessibility relation R^{Σ} between such “worlds.” We want it to be the case that the accessibility relation (and the assignment V^{Σ}) are defined in such a way that $\mathfrak{M}^{\Sigma}, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. How should we do this?

explanation

Once the accessibility relation is defined, the definition of truth at a world ensures that $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Box\varphi$ iff $\mathfrak{M}^{\Sigma}, \Delta' \Vdash \varphi$ for all Δ' such that $R^{\Sigma}\Delta\Delta'$. The proof that $\mathfrak{M}^{\Sigma}, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$ requires that this is true in particular for **formulas** starting with a modal operator, i.e., $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Box\varphi$ iff $\Box\varphi \in \Delta$. Combining this requirement with the definition of truth at a world for $\Box\varphi$ yields:

$$\Box\varphi \in \Delta \text{ iff } \varphi \in \Delta' \text{ for all } \Delta' \text{ with } R^{\Sigma}\Delta\Delta'$$

Consider the left-to-right direction: it says that if $\Box\varphi \in \Delta$, then $\varphi \in \Delta'$ for any φ and any Δ' with $R^{\Sigma}\Delta\Delta'$. If we stipulate that $R^{\Sigma}\Delta\Delta'$ iff $\varphi \in \Delta'$ for all $\Box\varphi \in \Delta$, then this holds. We can write the condition on the right of the “iff” more compactly as: $\{\varphi : \Box\varphi \in \Delta\} \subseteq \Delta'$.

So the question is: does this definition of R^{Σ} in fact guarantee that $\Box\varphi \in \Delta$ iff $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Box\varphi$? Does it also guarantee that $\Diamond\varphi \in \Delta$ iff $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Diamond\varphi$? The next few results will establish this.

Definition com.5. If Γ is a set of **formulas**, let

$$\begin{aligned}\Box\Gamma &= \{\Box\psi : \psi \in \Gamma\} \\ \Diamond\Gamma &= \{\Diamond\psi : \psi \in \Gamma\}\end{aligned}$$

and

$$\begin{aligned}\Box^{-1}\Gamma &= \{\psi : \Box\psi \in \Gamma\} \\ \Diamond^{-1}\Gamma &= \{\psi : \Diamond\psi \in \Gamma\}\end{aligned}$$

In other words, $\Box\Gamma$ is Γ with \Box in front of every **formula** in Γ ; $\Box^{-1}\Gamma$ is all the \Box 'ed **formulas** of Γ with the initial \Box 's removed. This definition is not terribly important on its own, but will simplify the notation considerably.

Note that $\Box\Box^{-1}\Gamma \subseteq \Gamma$:

$$\Box\Box^{-1}\Gamma = \{\Box\psi : \Box\psi \in \Gamma\}$$

i.e., it's just the set of all those **formulas** of Γ that start with \Box .

Lemma com.6. *If $\Gamma \vdash_{\Sigma} \varphi$ then $\Box\Gamma \vdash_{\Sigma} \Box\varphi$.*

*mod:com:mod:
lem:box1*

Proof. If $\Gamma \vdash_{\Sigma} \varphi$ then there are $\psi_1, \dots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \rightarrow (\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots)$. Since Σ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \rightarrow (\Box\psi_2 \rightarrow \dots (\Box\psi_n \rightarrow \Box\varphi) \dots)$, where obviously $\Box\psi_1, \dots, \Box\psi_k \in \Box\Gamma$. Hence, by definition, $\Box\Gamma \vdash_{\Sigma} \Box\varphi$. \square

Lemma com.7. *If $\Box^{-1}\Gamma \vdash_{\Sigma} \varphi$ then $\Gamma \vdash_{\Sigma} \Box\varphi$.*

*mod:com:mod:
lem:box2*

Proof. Suppose $\Box^{-1}\Gamma \vdash_{\Sigma} \varphi$; then by [Lemma com.6](#), $\Box\Box^{-1}\Gamma \vdash \Box\varphi$. But since $\Box\Box^{-1}\Gamma \subseteq \Gamma$, also $\Gamma \vdash_{\Sigma} \Box\varphi$ by Monotony. \square

Proposition com.8. *If Γ is complete Σ -consistent, then $\Box\varphi \in \Gamma$ if and only if for every complete Σ -consistent Δ such that $\Box^{-1}\Gamma \subseteq \Delta$, it holds that $\varphi \in \Delta$.*

*mod:com:mod:
prop:box*

Proof. Suppose Γ is complete Σ -consistent. The “only if” direction is easy: Suppose $\Box\varphi \in \Gamma$ and that $\Box^{-1}\Gamma \subseteq \Delta$. Since $\Box\varphi \in \Gamma$, $\varphi \in \Box^{-1}\Gamma \subseteq \Delta$, so $\varphi \in \Delta$.

For the “if” direction, we prove the contrapositive: Suppose $\Box\varphi \notin \Gamma$. Since Γ is complete Σ -consistent, it is deductively closed, and hence $\Gamma \not\vdash_{\Sigma} \Box\varphi$. By [Lemma com.7](#), $\Box^{-1}\Gamma \not\vdash_{\Sigma} \varphi$. By ?????, $\Box^{-1}\Gamma \cup \{\neg\varphi\}$ is Σ -consistent. By Lindenbaum’s Lemma, there is a complete Σ -consistent set Δ such that $\Box^{-1}\Gamma \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$. \square

Lemma com.9. *Suppose Γ and Δ are complete Σ -consistent. Then: $\Box^{-1}\Gamma \subseteq \Delta$ if and only if $\Diamond\Delta \subseteq \Gamma$.*

*mod:com:mod:
lem:box-iff-diamond*

Proof. “Only if” direction: Assume $\Box^{-1}\Gamma \subseteq \Delta$ and suppose $\Diamond\varphi \in \Diamond\Delta$ (i.e., $\varphi \in \Delta$). In order to show $\Diamond\varphi \in \Gamma$ it suffices to show $\Box\neg\varphi \notin \Gamma$ for then by maximality $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of Δ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

“If” direction: Assume $\Diamond\Delta \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in Γ , by consistency $\Box\varphi \notin \Gamma$, as required. \square

Proposition com.10. *If Γ is complete Σ -consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some complete Σ -consistent Δ such that $\Diamond\Delta \subseteq \Gamma$, it holds that $\varphi \in \Delta$.*

*mod:com:mod:
prop:diamond*

Proof. Suppose Γ is complete Σ -consistent. $\Diamond\varphi \in \Gamma$ iff $\neg\Box\neg\varphi \in \Gamma$ by DUAL and closure. $\neg\Box\neg\varphi \in \Gamma$ iff $\Box\neg\varphi \notin \Gamma$ by [Proposition com.2\(5\)](#) since Γ is complete Σ -consistent. By [Proposition com.8](#), $\Box\neg\varphi \notin \Gamma$ iff, for some complete Σ -consistent Δ with $\Box^{-1}\Gamma \subseteq \Delta$, $\neg\varphi \notin \Delta$. Now consider any such Δ . By [Lemma com.9](#), $\Box^{-1}\Gamma \subseteq \Delta$ iff $\Diamond\Delta \subseteq \Gamma$. Also, $\neg\varphi \notin \Delta$ iff $\varphi \in \Delta$ by [Proposition com.2\(5\)](#). So $\Diamond\varphi \in \Gamma$ iff, for some complete Σ -consistent Δ with $\Diamond\Delta \subseteq \Gamma$, $\varphi \in \Delta$. \square

Problem com.2. Show that if Γ is complete Σ -consistent, then $\Diamond\varphi \in \Gamma$ if and only if there is a complete Σ -consistent Δ such that $\Box^{-1}\Gamma \subseteq \Delta$ and $\varphi \in \Delta$. Do this without using [Lemma com.9](#).

com.5 Canonical Models

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The *canonical model* for a modal system Σ is a specific model \mathfrak{M}^Σ in which the worlds are all complete Σ -consistent sets. Its accessibility relation R^Σ and valuation V^Σ are defined so as to guarantee that the **formulas** true at a world Δ are exactly the **formulas** making up Δ .

Definition com.11. Let Σ be a normal modal logic. The *canonical model* for Σ is $\mathfrak{M}^\Sigma = \langle W^\Sigma, R^\Sigma, V^\Sigma \rangle$, where:

1. $\mathfrak{M}^\Sigma = \{\Delta : \Delta \text{ is complete } \Sigma\text{-consistent}\}$.
2. $R^\Sigma \Delta \Delta'$ holds if and only if $\Box^{-1} \Delta \subseteq \Delta'$.
3. $V^\Sigma(p) = \{\Delta : p \in \Delta\}$.

com.6 The Truth Lemma

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The canonical model \mathfrak{M}^Σ is defined in such a way that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. For propositional variables, the definition of V^Σ yields this directly. We have to verify that the equivalence holds for all **formulas**, however. We do this by induction. The inductive step involves proving the equivalence for **formulas** involving propositional operators (where we have to use [Proposition com.2](#)) and the modal operators (where we invoke the results of [section com.4](#)).

mod:com:tru:
prop:truthlemma

Proposition com.12 (Truth Lemma). *For every formula φ , $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ if and only if $\varphi \in \Delta$.*

Proof. By induction on φ .

1. $\varphi \equiv \perp$: $\mathfrak{M}^\Sigma, \Delta \not\Vdash \perp$ by ??, and $\perp \notin \Delta$ by [Proposition com.2\(3\)](#).
2. $\varphi \equiv \top$: $\mathfrak{M}^\Sigma, \Delta \Vdash \top$ by ??, and $\top \in \Delta$ by [Proposition com.2\(4\)](#).
3. $\varphi \equiv p$: $\mathfrak{M}^\Sigma, \Delta \Vdash p$ iff $\Delta \in V^\Sigma(p)$ by ?. Also, $\Delta \in V^\Sigma(p)$ iff $p \in \Delta$ by definition of V^Σ .
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \neg\psi$ iff $\mathfrak{M}^\Sigma, \Delta \not\Vdash \psi$ (??) iff $\psi \notin \Delta$ (by inductive hypothesis) iff $\neg\psi \in \Delta$ (by [Proposition com.2\(5\)](#)).
5. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \wedge \chi$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by ??) iff $\psi \in \Delta$ and $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \wedge \chi \in \Delta$ (by [Proposition com.2\(6\)](#)).
6. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \vee \chi$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ or $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by ??) iff $\psi \in \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \vee \chi \in \Delta$ (by [Proposition com.2\(7\)](#)).
7. $\varphi \equiv \psi \rightarrow \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \rightarrow \chi$ iff $\mathfrak{M}^\Sigma, \Delta \not\Vdash \psi$ or $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by ??) iff $\psi \notin \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \rightarrow \chi \in \Delta$ (by [Proposition com.2\(8\)](#)).

8. $\varphi \equiv \psi \leftrightarrow \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \leftrightarrow \chi$ iff either $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ or $\mathfrak{M}^\Sigma, \Delta \nVdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \nVdash \chi$ (by ??) iff either $\psi \in \Delta$ and $\chi \in \Delta$ or $\psi \notin \Delta$ and $\chi \notin \Delta$ (by inductive hypothesis) iff $\psi \leftrightarrow \chi \in \Delta$ (by [Proposition com.2\(9\)](#)).
9. $\varphi \equiv \Box\psi$: First suppose that $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$. By ??, for every Δ' such that $R^\Sigma \Delta \Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By inductive hypothesis, for every Δ' such that $R^\Sigma \Delta \Delta'$, $\psi \in \Delta'$. By definition of R^Σ , for every Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$, $\psi \in \Delta'$. By [Proposition com.8](#), $\Box\psi \in \Delta$.
Now assume $\Box\psi \in \Delta$. Let $\Delta' \in W^\Sigma$ be such that $R^\Sigma \Delta \Delta'$, i.e., $\Box^{-1}\Delta \subseteq \Delta'$. Since $\Box\psi \in \Delta$, $\psi \in \Box^{-1}\Delta$. Consequently, $\psi \in \Delta'$. By inductive hypothesis, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. Since Δ' is arbitrary with $R^\Sigma \Delta \Delta'$, for all $\Delta' \in W^\Sigma$ such that $R^\Sigma \Delta \Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By ??, $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$.
10. $\varphi \equiv \Diamond\psi$: First suppose that $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\psi$. By ??, for some Δ' such that $R^\Sigma \Delta \Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By inductive hypothesis, for some Δ' such that $R^\Sigma \Delta \Delta'$, $\psi \in \Delta'$. By definition of R^Σ , for some Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$, $\psi \in \Delta'$. By [Proposition com.10](#), for some Δ' such that $\Diamond\Delta' \subseteq \Delta$, $\psi \in \Delta'$. Since $\psi \in \Delta'$, $\Diamond\psi \in \Diamond\Delta'$, so $\Diamond\psi \in \Delta$.
Now assume $\Diamond\psi \in \Delta$. By [Proposition com.10](#), there is a complete Σ -consistent $\Delta' \in W^\Sigma$ be such that $\Diamond\Delta' \subseteq \Delta$ and $\psi \in \Delta'$. By [Lemma com.9](#), there is a $\Delta' \in W^\Sigma$ such that $\Box^{-1}\Delta \subseteq \Delta'$, and $\psi \in \Delta'$. By definition of R^Σ , $R^\Sigma \Delta \Delta'$, so there is a $\Delta' \in W^\Sigma$ such that $R^\Sigma \Delta \Delta'$ and $\psi \in \Delta'$. By ??, $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\psi$.

□

Problem com.3. Complete the proof of [Proposition com.12](#).

com.7 Determination and Completeness for K

We are now prepared to use the canonical model to establish completeness. Completeness follows from the fact that the [formulas](#) true in the canonical for Σ are exactly the Σ -[derivable](#) ones. Models with this property are said to *determine* Σ . mod:com:cmk:
sec

Definition com.13. A model \mathfrak{M} *determines* a normal modal logic Σ precisely when $\mathfrak{M} \Vdash \varphi$ if and only if $\Sigma \vdash \varphi$, for all [formulas](#) φ .

Theorem com.14 (Determination). $\mathfrak{M}^\Sigma \Vdash \varphi$ if and only if $\Sigma \vdash \varphi$. mod:com:cmk:
thm:determination

Proof. If $\mathfrak{M}^\Sigma \Vdash \varphi$, then for every complete Σ -consistent Δ , we have $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$. Hence, by the Truth Lemma, $\varphi \in \Delta$ for every complete Σ -consistent Δ , whence by [Corollary com.4](#) (with $\Gamma = \emptyset$), $\Sigma \vdash \varphi$.

Conversely, if $\Sigma \vdash \varphi$ then by [Proposition com.2\(1\)](#), every complete Σ -consistent Δ contains φ , and hence by the Truth Lemma, $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ for every $\Delta \in W^\Sigma$, i.e., $\mathfrak{M}^\Sigma \Vdash \varphi$. □

Since the canonical model for \mathbf{K} determines \mathbf{K} , we immediately have completeness of \mathbf{K} as a corollary:

mod:com:cmk: **Corollary com.15.** *cor:Kcomplete* The basic modal logic \mathbf{K} is complete with respect to the class of all models, i.e., if $\models \varphi$ then $\mathbf{K} \vdash \varphi$.

Proof. Contrapositively, if $\mathbf{K} \not\vdash \varphi$ then by Determination $\mathfrak{M}^{\mathbf{K}} \not\models \varphi$ and hence φ is not valid. \square

For the general case of completeness of a system Σ with respect to a class of models, e.g., of $\mathbf{KTB4}$ with respect to the class of reflexive, symmetric, transitive models, determination alone is not enough. We must also show that the canonical model for the system Σ is a member of the class, which does not follow obviously from the canonical model construction—nor is it always true!

com.8 Frame Completeness

mod:com:fra: The completeness theorem for \mathbf{K} can be extended to other modal systems, *sec* once we show that the canonical model for a given logic has the corresponding frame property.

mod:com:fra: **Theorem com.16.** *thm:completeframeprops* If a normal modal logic Σ contains one of the formulas on the left-hand side of table com.1, then the canonical model for Σ has the corresponding property on the right-hand side.

If Σ contains the canonical model for Σ is:
D: $\Box\varphi \rightarrow \Diamond\varphi$	serial;
T: $\Box\varphi \rightarrow \varphi$	reflexive;
B: $\varphi \rightarrow \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \rightarrow \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$	euclidean.

Table com.1: Basic correspondence facts.

mod:com:fra:
tab:correspondencetable

Proof. We take each of these up in turn.

Suppose Σ contains D, and let $\Delta \in W^\Sigma$; we need to show that there is a Δ' such that $R^\Sigma \Delta \Delta'$. It suffices to show that $\Box^{-1}\Delta$ is Σ -consistent, for then by Lindenbaum's Lemma, there is a complete Σ -consistent set $\Delta' \supseteq \Box^{-1}\Delta$, and by definition of R^Σ we have $R^\Sigma \Delta \Delta'$. So, suppose for contradiction that $\Box^{-1}\Delta$ is *not* Σ -consistent, i.e., $\Box^{-1}\Delta \vdash_\Sigma \perp$. By Lemma com.7, $\Delta \vdash_\Sigma \Box\perp$, and since Σ contains D, also $\Delta \vdash_\Sigma \Diamond\perp$. But Σ is normal, so $\Sigma \vdash \neg\Diamond\perp$ (?), whence also $\Delta \vdash_\Sigma \neg\Diamond\perp$, against the consistency of Δ .

Now suppose Σ contains T, and let $\Delta \in W^\Sigma$. We want to show $R^\Sigma \Delta \Delta$, i.e., $\Box^{-1}\Delta \subseteq \Delta$. But if $\Box\varphi \in \Delta$ then by T also $\varphi \in \Delta$, as desired.

Now suppose Σ contains B, and suppose $R^\Sigma \Delta \Delta'$ for $\Delta, \Delta' \in W^\Sigma$. We need to show that $R^\Sigma \Delta' \Delta$, i.e., $\Box^{-1}\Delta' \subseteq \Delta$. By Lemma com.9, this is equivalent

to $\diamond\Delta \subseteq \Delta'$. So suppose $\varphi \in \Delta$. By B, also $\Box\diamond\varphi \in \Delta$. By the hypothesis that $R^\Sigma\Delta\Delta'$, we have that $\Box^{-1}\Delta \subseteq \Delta'$, and hence $\diamond\varphi \in \Delta'$, as required.

Now suppose Σ contains 4, and suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_2\Delta_3$. We need to show $R^\Sigma\Delta_1\Delta_3$. From the hypothesis we have both $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and $\Box^{-1}\Delta_2 \subseteq \Delta_3$. In order to show $R^\Sigma\Delta_1\Delta_3$ it suffices to show $\Box^{-1}\Delta_1 \subseteq \Delta_3$. So let $\psi \in \Box^{-1}\Delta_1$, i.e., $\Box\psi \in \Delta_1$. By 4, also $\Box\Box\psi \in \Delta_1$ and by hypothesis we get, first, that $\Box\psi \in \Delta_2$ and, second, that $\psi \in \Delta_3$, as desired.

Now suppose Σ contains 5, suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$. We need to show $R^\Sigma\Delta_2\Delta_3$. The first hypothesis gives $\Box^{-1}\Delta_1 \subseteq \Delta_2$, and the second hypothesis is equivalent to $\diamond\Delta_3 \subseteq \Delta_2$, by [Lemma com.9](#). To show $R^\Sigma\Delta_2\Delta_3$, by [Lemma com.9](#), it suffices to show $\diamond\Delta_3 \subseteq \Delta_2$. So let $\diamond\varphi \in \diamond\Delta_3$, i.e., $\varphi \in \Delta_3$. By the second hypothesis $\diamond\varphi \in \Delta_1$ and by 5, $\Box\diamond\varphi \in \Delta_1$ as well. But now the first hypothesis gives $\diamond\varphi \in \Delta_2$, as desired. \square

As a corollary we obtain completeness results for a number of systems. For instance, we know that $\mathbf{S5} = \mathbf{KT5} = \mathbf{KTB4}$ is complete with respect to the class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

Theorem com.17. *Let $\mathcal{C}_D, \mathcal{C}_T, \mathcal{C}_B, \mathcal{C}_4$, and \mathcal{C}_5 be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1, \dots, \varphi_n$ among D, T, B, 4, and 5, the system $\mathbf{K}\varphi_1 \dots \varphi_n$ is determined by the class of models $\mathcal{C} = \mathcal{C}_{\varphi_1} \cap \dots \cap \mathcal{C}_{\varphi_n}$.*

*mod:com:fra:
thm:generaldet*

Proposition com.18. *Let Σ be a normal modal logic; then:*

1. *If Σ contains the schema $\diamond\varphi \rightarrow \Box\varphi$ then the canonical model for Σ is partially functional.*
2. *If Σ contains the schema $\diamond\varphi \leftrightarrow \Box\varphi$ then the canonical model for Σ is functional.*
3. *If Σ contains the schema $\Box\Box\varphi \rightarrow \Box\varphi$ then the canonical model for Σ is weakly dense.*

*mod:com:fra:
prop:anotherfive-a*

(see ?? for definitions of these frame properties).

Proof. 1. suppose that Σ contains the schema $\diamond\varphi \rightarrow \Box\varphi$, to show that R^Σ is partially functional we need to prove that for any $\Delta_1, \Delta_2, \Delta_3 \in W^\Sigma$, if $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$ then $\Delta_2 = \Delta_3$. Since $R^\Sigma\Delta_1\Delta_2$ we have $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and since $R^\Sigma\Delta_1\Delta_3$ also $\Box^{-1}\Delta_1 \subseteq \Delta_3$. The identity $\Delta_2 = \Delta_3$ will follow if we can establish the two inclusions $\Delta_2 \subseteq \Delta_3$ and $\Delta_3 \subseteq \Delta_2$. For the first inclusion, let $\varphi \in \Delta_2$; then $\diamond\varphi \in \Delta_1$, and by the schema and deductive closure of Δ_1 also $\Box\varphi \in \Delta_1$, whence by the hypothesis that $R^\Sigma\Delta_1\Delta_3$, $\varphi \in \Delta_3$. The second inclusion is similar.

2. This follows immediately from part (1) and the seriality proof in [Theorem com.16](#).

3. Suppose Σ contains the schema $\Box\Box\varphi \rightarrow \Box\varphi$ and to show that R^Σ is weakly dense, let $R^\Sigma\Delta_1\Delta_2$. We need to show that there is a complete Σ -consistent set Δ_3 such that $R^\Sigma\Delta_1\Delta_3$ and $R^\Sigma\Delta_3\Delta_2$. Let:

$$\Gamma = \Box^{-1}\Delta_1 \cup \Diamond\Delta_2.$$

It suffices to show that Γ is Σ -consistent, for then by Lindenbaum's Lemma it can be extended to a complete Σ -consistent set Δ_3 such that $\Box^{-1}\Delta_1 \subseteq \Delta_3$ and $\Diamond\Delta_2 \subseteq \Delta_3$, i.e., $R^\Sigma\Delta_1\Delta_3$ and $R^\Sigma\Delta_3\Delta_2$ (by [Lemma com.9](#)).

Suppose for contradiction that Γ is not consistent. Then there are formulas $\Box\varphi_1, \dots, \Box\varphi_n \in \Delta_1$ and $\psi_1, \dots, \psi_m \in \Delta_2$ such that

$$\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_\Sigma \perp.$$

Since $\Diamond(\psi_1 \wedge \dots \wedge \psi_m) \rightarrow (\Diamond\psi_1 \wedge \dots \wedge \Diamond\psi_m)$ is [derivable](#) in every normal modal logic, we argue as follows, contradicting the consistency of Δ_2 :

$$\begin{aligned} & \varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_\Sigma \perp \\ & \Rightarrow \varphi_1, \dots, \varphi_n \vdash_\Sigma (\Diamond\psi_1 \wedge \dots \wedge \Diamond\psi_m) \rightarrow \perp, & \text{deduction theorem;} \\ & \Rightarrow \varphi_1, \dots, \varphi_n \vdash_\Sigma \Diamond(\psi_1 \wedge \dots \wedge \psi_m) \rightarrow \perp, & \Sigma \text{ is normal;} \\ & \Rightarrow \varphi_1, \dots, \varphi_n \vdash_\Sigma \Box\neg(\psi_1 \wedge \dots \wedge \psi_m), & \text{PL;} \\ & \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n \vdash_\Sigma \Box\Box\neg(\psi_1 \wedge \dots \wedge \psi_m), & \text{Lemma com.6;} \\ & \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n \vdash_\Sigma \Box\neg(\psi_1 \wedge \dots \wedge \psi_m), & \text{by the schema;} \\ & \Rightarrow \Delta_1 \vdash_\Sigma \Box\neg(\psi_1 \wedge \dots \wedge \psi_m), & \text{Monotony;} \\ & \Rightarrow \Box\neg(\psi_1 \wedge \dots \wedge \psi_m) \in \Delta_1, & \text{deductive closure;} \\ & \Rightarrow \neg(\psi_1 \wedge \dots \wedge \psi_m) \in \Delta_2, & \text{since } R^\Sigma\Delta_1\Delta_2. \end{aligned}$$

□

On the strength of these examples, one might think that every system Σ of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of Σ is valid. Unfortunately, there are many systems that are not complete in this sense.

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Bibliography