Chapter udf

Completeness and Canonical Models

com.1 Introduction

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If Σ is a modal system, then the soundness theorem establishes that if $\Sigma \vdash \varphi$, then φ is valid in any class $\mathcal{C}$ of models in which all instances of all formulas in Σ are valid. In particular that means that if $\mathbf{K} \vdash \varphi$ then φ is true in all models; if $\mathbf{KT} \vdash \varphi$ then φ is true in all reflexive models; if $\mathbf{KD} \vdash \varphi$ then φ is true in all serial models, etc.

Completeness is the converse of soundness: that $\mathbf{K}$ is complete means that if a formula φ is valid, $\vdash \varphi$, for instance. Proving completeness is a lot harder to do than proving soundness. It is useful, first, to consider the contrapositive: $\mathbf{K}$ is complete iff whenever $\nvdash \varphi$, there is a countermodel, i.e., a model $\mathfrak{M}$ such that $\mathfrak{M} \nVdash \varphi$. Equivalently (negating φ), we could prove that whenever $\nvdash \neg\varphi$, there is a model of φ. In the construction of such a model, we can use information contained in φ. When we find models for specific formulas we often do the same: e.g., if we want to find a countermodel to $p \rightarrow \Box q$, we know that it has to contain a world where p is true and $\Box q$ is false. And a world where $\Box q$ is false means there has to be a world accessible from it where q is false. And that's all we need to know: which worlds make the propositional variables true, and which worlds are accessible from which worlds.

In the case of proving completeness, however, we don't have a specific formula φ for which we are constructing a model. We want to establish that a model exists for every φ such that $\nvdash_\Sigma \neg\varphi$. This is a minimal requirement, since *if* $\vdash_\Sigma \neg\varphi$, by soundness, there is no model for φ (in which Σ is true). Now note that $\nvdash_\Sigma \neg\varphi$ iff φ is Σ-consistent. (Recall that $\Sigma \nvdash_\Sigma \neg\varphi$ and $\varphi \nvdash_\Sigma \bot$ are equivalent.) So our task is to construct a model for every Σ-consistent formula.

The trick we'll use is to find a Σ-consistent set of formulas that contains φ, but also other formulas which tell us what the world that makes φ true has to look like. Such sets are *complete* Σ-consistent sets. It's not enough to construct a model with a single world to make φ true, it will have to contain multiple worlds and an accessibility relation. The complete Σ-consistent set containing φ will also contain other formulas of the form $\Box\psi$ and $\Diamond\chi$. In all accessible worlds, ψ has to be true; in at least one, χ has to be true. In order to accomplish this, we'll simply take *all* possible complete Σ-consistent sets as the basis for the set of worlds. A tricky part will be to figure out when a complete Σ-consistent set should count as being accessible from another in our model.

We'll show that in the model so defined, φ is true at a world—which is also a complete Σ-consistent set—iff φ is an element of that set. If φ is Σ-consistent, it will be an element of at least one complete Σ-consistent set (a fact we'll prove), and so there will be a world where φ is true. So we will have a single model where every Σ-consistent formula φ is true at some world. This single model is the *canonical* model for Σ.

com.2 Complete Σ-Consistent Sets

Suppose Σ is a set of modal formulas—think of them as the axioms or defining principles of a normal modal logic. A set Γ is Σ-consistent iff $\Gamma \nvdash_{\Sigma} \bot$, i.e., if there is no derivation of $\varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_n \rightarrow \bot) \dots)$ from Σ, where each $\varphi_i \in \Gamma$. We will construct a "canonical" model in which each world is taken to be a special kind of Σ-consistent set: one which is not just Σ-consistent, but maximally so, in the sense that it settles the truth value of every modal formula: for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$: nml:com:ccs:sec

Definition com.1. A set Γ is *complete Σ-consistent* if and only if it is Σ-consistent and for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Complete Σ-consistent sets Γ have a number of useful properties. For one, they are deductively closed, i.e., if $\Gamma \vdash_{\Sigma} \varphi$ then $\varphi \in \Gamma$. This means in particular that every instance of a formula $\varphi \in \Sigma$ is also $\in \Gamma$. Moreover, membership in Γ mirrors the truth conditions for the propositional connectives. This will be important when we define the "canonical model."

Proposition com.2. *Suppose Γ is complete Σ-consistent. Then:* *nml:com:ccs:prop:ccs-properties*

1. *Γ is deductively closed in Σ.* *nml:com:ccs:prop:ccs-closed*
2. *$\Sigma \subseteq \Gamma$.* *nml:com:ccs:prop:ccs-sigma*
3. *$\bot \notin \Gamma$* *nml:com:ccs:prop:ccs-lfalse*
4. *$\top \in \Gamma$* *nml:com:ccs:prop:ccs-ltrue*
5. *$\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.* *nml:com:ccs:prop:ccs-lnot*
6. *$\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$* *nml:com:ccs:prop:ccs-land*
7. *$\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$* *nml:com:ccs:prop:ccs-lor*

nml:com:ccs: prop:ccs-lif

8. *$\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$*

nml:com:ccs: prop:ccs-liff

9. *$\varphi \leftrightarrow \psi \in \Gamma$ iff either $\varphi \in \Gamma$ and $\psi \in \Gamma$, or $\varphi \notin \Gamma$ and $\psi \notin \Gamma$*

Proof. 1. Suppose $\Gamma \vdash_\Sigma \varphi$ but $\varphi \notin \Gamma$. Then since Γ is complete Σ-consistent, $\neg\varphi \in \Gamma$. This would make Γ inconsistent, since $\varphi, \neg\varphi \vdash_\Sigma \bot$.

2. If $\varphi \in \Sigma$ then $\Gamma \vdash_\Sigma \varphi$, and $\varphi \in \Gamma$ by deductive closure, i.e., case (1).

3. If $\bot \in \Gamma$, then $\Gamma \vdash_\Sigma \bot$, so Γ would be Σ-inconsistent.

4. $\Gamma \vdash_\Sigma \top$, so $\top \in \Gamma$ by deductive closure, i.e., case (1).

5. If $\neg\varphi \in \Gamma$, then by consistency $\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then $\varphi \in \Gamma$ since Γ is complete Σ-consistent.

6. Suppose $\varphi \wedge \psi \in \Gamma$. Since $(\varphi \wedge \psi) \to \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure, i.e., case (1). Similarly for $\psi \in \Gamma$. On the other hand, suppose both $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then deductive closure implies $(\varphi \wedge \psi) \in \Gamma$, since $\varphi \to (\psi \to (\varphi \wedge \psi))$ is a tautological instance.

7. Suppose $\varphi \vee \psi \in \Gamma$, and $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is complete Σ-consistent, $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. Then $\neg(\varphi \vee \psi) \in \Gamma$ since $\neg\varphi \to (\neg\psi \to \neg(\varphi \vee \psi))$ is a tautological instance. This would mean that Γ is Σ-inconsistent, a contradiction.

8. Suppose $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash_\Sigma \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \to \psi \notin \Gamma$ then since Γ is complete Σ-consistent, $\neg(\varphi \to \psi) \in \Gamma$. Since $\neg(\varphi \to \psi) \to \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure. Since $\neg(\varphi \to \psi) \to \neg\psi$ is a tautological instance, $\neg\psi \in \Gamma$. Then $\psi \notin \Gamma$ since Γ is Σ-consistent.

9. Suppose $\varphi \leftrightarrow \psi \in \Gamma$. If $\varphi \in \Gamma$, then $\psi \in \Gamma$, since $(\varphi \leftrightarrow \psi) \to (\varphi \to \psi)$ is a tautological instance. Similarly, if $\psi \in \Gamma$, then $\varphi \in \Gamma$. So either both $\varphi \in \Gamma$ and $\psi \in \Gamma$, or neither $\varphi \in \Gamma$ nor $\psi \in \Gamma$.

   Conversely, suppose $\varphi \to \psi \notin \Gamma$. Since Γ is complete Σ-consistent, $\neg(\varphi \leftrightarrow \psi) \in \Gamma$. Since $\neg(\varphi \leftrightarrow \psi) \to (\varphi \to \neg\psi)$ is a tautological instance, if $\varphi \in \Gamma$ then $\neg\psi \in \Gamma$, and since Γ is Σ-consistent, $\psi \notin \Gamma$. Similarly, if $\psi \in \Gamma$ then $\varphi \notin \Gamma$. So neither $\varphi \in \Gamma$ and $\psi \in \Gamma$, nor $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. □

Problem com.1. Complete the proof of Proposition com.2.

com.3 Lindenbaum's Lemma

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Lindenbaum's Lemma establishes that every Σ-consistent set of formulas is contained in at least one *complete* Σ-consistent set. Our construction of the canonical model will show that for each complete Σ-consistent set Δ, there is a world in the canonical model where all and only the formulas in Δ are true. So

Lindenbaum's Lemma guarantees that every Σ-consistent set is true at some world in the canonical model.

Theorem com.3 (Lindenbaum's Lemma). *If Γ is Σ-consistent then there is a complete Σ-consistent set Δ extending Γ.* nml:com:lin:thm:lindenbaum

Proof. Let $\varphi_0, \varphi_1, \dots$ be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing p_0, and at each stage $n \geq 1$ list the finitely many formulas of length n using only variables among $p_0, \dots, p_n$. We define sets of formulas Δ_n by induction on n, and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$. Supposing that Δ_n has been defined, we define Δ_{n+1} by:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is } \Sigma\text{-consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

Now let $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$.

We have to show that this definition actually yields a set Δ with the required properties, i.e., $\Gamma \subseteq \Delta$ and Δ is complete Σ-consistent.

It's obvious that $\Gamma \subseteq \Delta$, since $\Delta_0 \subseteq \Delta$ by construction, and $\Delta_0 = \Gamma$. In fact, $\Delta_n \subseteq \Delta$ for all n, since Δ is the union of all Δ_n. (Since in each step of the construction, we add a formula to the set already constructed, $\Delta_n \subseteq \Delta_{n+1}$, so since $\subseteq$ is transitive, $\Delta_n \subseteq \Delta_m$ whenever $n \leq m$.) At each stage of the construction, we either add φ_n or $\neg\varphi_n$, and every formula appears (at least once) in the list of all φ_n. So, for every φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$, so Δ is complete by definition.

Finally, we have to show, that Δ is Σ-consistent. To do this, we show that (a) if Δ were Σ-inconsistent, then some Δ_n would be Σ-inconsistent, and (b) all Δ_n are Σ-consistent.

So suppose Δ were Σ-inconsistent. Then $\Delta \vdash_\Sigma \bot$, i.e., there are φ_1, $\dots, \varphi_k \in \Delta$ such that $\Sigma \vdash \varphi_1 \to (\varphi_2 \to \cdots (\varphi_k \to \bot) \dots)$. Since $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$, each $\varphi_i \in \Delta_{n_i}$ for some n_i. Let n be the largest of these. Since $n_i \leq n$, $\Delta_{n_i} \subseteq \Delta_n$. So, all φ_i are in some Δ_n. This would mean $\Delta_n \vdash_\Sigma \bot$, i.e., Δ_n is Σ-inconsistent.

To show that each Δ_n is Σ-consistent, we use a simple induction on n. $\Delta_0 = \Gamma$, and we assumed Γ was Σ-consistent. So the claim holds for $n = 0$. Now suppose it holds for n, i.e., Δ_n is Σ-consistent. Δ_{n+1} is either $\Delta_n \cup \{\varphi_n\}$ if that is Σ-consistent, otherwise it is $\Delta_n \cup \{\neg\varphi_n\}$. In the first case, Δ_{n+1} is clearly Σ-consistent. However, by **????**, either $\Delta_n \cup \{\varphi_n\}$ or $\Delta_n \cup \{\neg\varphi_n\}$ is consistent, so Δ_{n+1} is consistent in the other case as well. □

Corollary com.4. *$\Gamma \vdash_\Sigma \varphi$ if and only if $\varphi \in \Delta$ for each complete Σ-consistent set Δ extending Γ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system Σ.)* nml:com:lin:cor:provability-characterization

Proof. Suppose $\Gamma \vdash_\Sigma \varphi$, and let Δ be any complete Σ-consistent set extending Γ. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_\Sigma \varphi$ (by monotonicity) and

$\Delta \vdash_\Sigma \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \nvdash_\Sigma \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent, and by Lindenbaum's Lemma there is a complete consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$. By consistency, $\varphi \notin \Delta$. □

com.4 Modalities and Complete Consistent Sets

nml:com:mod:sec When we construct a model $\mathfrak{M}^\Sigma$ whose set of worlds is given by the complete Σ-consistent sets Δ in some normal modal logic Σ, we will also need to define an accessibility relation R^Σ between such "worlds." We want it to be the case that the accessibility relation (and the assignment V^Σ) are defined in such a way that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. How should we do this? explanation

Once the accessibility relation is defined, the definition of truth at a world ensures that $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$ iff $\mathfrak{M}^\Sigma, \Delta' \Vdash \varphi$ for all Δ' such that $R^\Sigma\Delta\Delta'$. The proof that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$ requires that this is true in particular for formulas starting with a modal operator, i.e., $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$ iff $\Box\varphi \in \Delta$. Combining this requirement with the definition of truth at a world for $\Box\varphi$ yields:

$$\Box\varphi \in \Delta \text{ iff } \varphi \in \Delta' \text{ for all } \Delta' \text{ with } R^\Sigma\Delta\Delta'$$

Consider the left-to-right direction: it says that if $\Box\varphi \in \Delta$, then $\varphi \in \Delta'$ for any φ and any Δ' with $R^\Sigma\Delta\Delta'$. If we stipulate that $R^\Sigma\Delta\Delta'$ iff $\varphi \in \Delta'$ for all $\Box\varphi \in \Delta$, then this holds. We can write the condition on the right of the "iff" more compactly as: $\{\varphi : \Box\varphi \in \Delta\} \subseteq \Delta'$.

So the question is: does this definition of R^Σ in fact guarantee that $\Box\varphi \in \Delta$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$? Does it also guarantee that $\Diamond\varphi \in \Delta$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\varphi$? The next few results will establish this.

Definition com.5. If Γ is a set of formulas, let

$$\Box\Gamma = \{\Box\psi : \psi \in \Gamma\}$$
$$\Diamond\Gamma = \{\Diamond\psi : \psi \in \Gamma\}$$

and

$$\Box^{-1}\Gamma = \{\psi : \Box\psi \in \Gamma\}$$
$$\Diamond^{-1}\Gamma = \{\psi : \Diamond\psi \in \Gamma\}$$

In other words, $\Box\Gamma$ is Γ with $\Box$ in front of every formula in Γ; $\Box^{-1}\Gamma$ is all the $\Box$'ed formulas of Γ with the initial $\Box$'s removed. This definition is not terribly important on its own, but will simplify the notation considerably.

Note that $\Box\Box^{-1}\Gamma \subseteq \Gamma$:

$$\Box\Box^{-1}\Gamma = \{\Box\psi : \Box\psi \in \Gamma\}$$

i.e., it's just the set of all those formulas of Γ that start with $\Box$.

Lemma com.6. *If $\Gamma \vdash_\Sigma \varphi$ then $\Box\Gamma \vdash_\Sigma \Box\varphi$.* nml:com:mod:lem:box1

Proof. If $\Gamma \vdash_\Sigma \varphi$ then there are $\psi_1, \dots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$. Since Σ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \to (\Box\psi_2 \to \cdots (\Box\psi_n \to \Box\varphi) \cdots)$, where obviously $\Box\psi_1, \dots, \Box\psi_k \in \Box\Gamma$. Hence, by definition, $\Box\Gamma \vdash_\Sigma \Box\varphi$. □

Lemma com.7. *If $\Box^{-1}\Gamma \vdash_\Sigma \varphi$ then $\Gamma \vdash_\Sigma \Box\varphi$.* nml:com:mod:lem:box2

Proof. Suppose $\Box^{-1}\Gamma \vdash_\Sigma \varphi$; then by Lemma com.6, $\Box\Box^{-1}\Gamma \vdash \Box\varphi$. But since $\Box\Box^{-1}\Gamma \subseteq \Gamma$, also $\Gamma \vdash_\Sigma \Box\varphi$ by monotonicity. □

Proposition com.8. *If Γ is complete Σ-consistent, then $\Box\varphi \in \Gamma$ if and only if for every complete Σ-consistent Δ such that $\Box^{-1}\Gamma \subseteq \Delta$, it holds that $\varphi \in \Delta$.* nml:com:mod:prop:box

Proof. Suppose Γ is complete Σ-consistent. The "only if" direction is easy: Suppose $\Box\varphi \in \Gamma$ and that $\Box^{-1}\Gamma \subseteq \Delta$. Since $\Box\varphi \in \Gamma$, $\varphi \in \Box^{-1}\Gamma \subseteq \Delta$, so $\varphi \in \Delta$.

For the "if" direction, we prove the contrapositive: Suppose $\Box\varphi \notin \Gamma$. Since Γ is complete Σ-consistent, it is deductively closed, and hence $\Gamma \nvdash_\Sigma \Box\varphi$. By Lemma com.7, $\Box^{-1}\Gamma \nvdash_\Sigma \varphi$. By **????**, $\Box^{-1}\Gamma \cup \{\neg\varphi\}$ is Σ-consistent. By Lindenbaum's Lemma, there is a complete Σ-consistent set Δ such that $\Box^{-1}\Gamma \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$. □

Lemma com.9. *Suppose Γ and Δ are complete Σ-consistent. Then $\Box^{-1}\Gamma \subseteq \Delta$ if and only if $\Diamond\Delta \subseteq \Gamma$.* nml:com:mod:lem:box-iff-diamond

Proof. "Only if" direction: Assume $\Box^{-1}\Gamma \subseteq \Delta$ and suppose $\Diamond\varphi \in \Diamond\Delta$ (i.e., $\varphi \in \Delta$). In order to show $\Diamond\varphi \in \Gamma$, it suffices to show $\Box\neg\varphi \notin \Gamma$, for then by maximality, $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of Δ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

"If" direction: Assume $\Diamond\Delta \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in Γ, by consistency $\Box\varphi \notin \Gamma$, as required. □

Proposition com.10. *If Γ is complete Σ-consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some complete Σ-consistent Δ such that $\Diamond\Delta \subseteq \Gamma$, it holds that $\varphi \in \Delta$.* nml:com:mod:prop:diamond

Proof. Suppose Γ is complete Σ-consistent. $\Diamond\varphi \in \Gamma$ iff $\neg\Box\neg\varphi \in \Gamma$ by DUAL and closure. $\neg\Box\neg\varphi \in \Gamma$ iff $\Box\neg\varphi \notin \Gamma$ by Proposition com.2(5) since Γ is complete Σ-consistent. By Proposition com.8, $\Box\neg\varphi \notin \Gamma$ iff, for some complete Σ-consistent Δ with $\Box^{-1}\Gamma \subseteq \Delta$, $\neg\varphi \notin \Delta$. Now consider any such Δ. By Lemma com.9, $\Box^{-1}\Gamma \subseteq \Delta$ iff $\Diamond\Delta \subseteq \Gamma$. Also, $\neg\varphi \notin \Delta$ iff $\varphi \in \Delta$ by Proposition com.2(5). So $\Diamond\varphi \in \Gamma$ iff, for some complete Σ-consistent Δ with $\Diamond\Delta \subseteq \Gamma$, $\varphi \in \Delta$. □

Problem com.2. Show that if Γ is complete Σ-consistent, then $\Diamond\varphi \in \Gamma$ if and only if there is a complete Σ-consistent Δ such that $\Box^{-1}\Gamma \subseteq \Delta$ and $\varphi \in \Delta$. *Do this without using Lemma com.9.*

com.5 Canonical Models

nml:com:cmd: sec The *canonical model* for a modal system Σ is a specific model $\mathfrak{M}^\Sigma$ in which the worlds are all complete Σ-consistent sets. Its accessibility relation R^Σ and valuation V^Σ are defined so as to guarantee that the formulas true at a world Δ are exactly the formulas making up Δ.

Definition com.11. Let Σ be a normal modal logic. The *canonical model* for Σ is $\mathfrak{M}^\Sigma = \langle W^\Sigma, R^\Sigma, V^\Sigma \rangle$, where:

1. $W^\Sigma = \{\Delta : \Delta \text{ is complete } \Sigma\text{-consistent}\}$.
2. $R^\Sigma \Delta\Delta'$ holds if and only if $\Box^{-1}\Delta \subseteq \Delta'$.
3. $V^\Sigma(p) = \{\Delta : p \in \Delta\}$.

com.6 The Truth Lemma

nml:com:tru: sec The canonical model $\mathfrak{M}^\Sigma$ is defined in such a way that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. For propositional variables, the definition of V^Σ yields this directly. We have to verify that the equivalence holds for all formulas, however. We do this by induction. The inductive step involves proving the equivalence for formulas involving propositional operators (where we have to use Proposition com.2) and the modal operators (where we invoke the results of section com.4).

nml:com:tru: prop:truthlemma **Proposition com.12 (Truth Lemma).** *For every formula φ, $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ if and only if $\varphi \in \Delta$.*

Proof. By induction on φ.

1. $\varphi \equiv \bot$: $\mathfrak{M}^\Sigma, \Delta \nVdash \bot$ by **??**, and $\bot \notin \Delta$ by Proposition com.2(3).
2. $\varphi \equiv \top$: $\mathfrak{M}^\Sigma, \Delta \Vdash \top$ by **??**, and $\top \in \Delta$ by Proposition com.2(4).
3. $\varphi \equiv p$: $\mathfrak{M}^\Sigma, \Delta \Vdash p$ iff $\Delta \in V^\Sigma(p)$ by **??**. Also, $\Delta \in V^\Sigma(p)$ iff $p \in \Delta$ by definition of V^Σ.
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \neg\psi$ iff $\mathfrak{M}^\Sigma, \Delta \nVdash \psi$ (**??**) iff $\psi \notin \Delta$ (by inductive hypothesis) iff $\neg\psi \in \Delta$ (by Proposition com.2(5)).
5. $\varphi \equiv \psi \land \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \land \chi$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \in \Delta$ and $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \land \chi \in \Delta$ (by Proposition com.2(6)).

6. $\varphi \equiv \psi \lor \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \lor \chi$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ or $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \in \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \lor \chi \in \Delta$ (by Proposition com.2(7)).

7. $\varphi \equiv \psi \to \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \to \chi$ iff $\mathfrak{M}^\Sigma, \Delta \nVdash \psi$ or $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \notin \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \to \chi \in \Delta$ (by Proposition com.2(8)).

8. $\varphi \equiv \psi \leftrightarrow \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \leftrightarrow \chi$ iff either $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ or $\mathfrak{M}^\Sigma, \Delta \nVdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \nVdash \chi$ (by **??**) iff either $\psi \in \Delta$ and $\chi \in \Delta$ or $\psi \notin \Delta$ and $\chi \notin \Delta$ (by inductive hypothesis) iff $\psi \leftrightarrow \chi \in \Delta$ (by Proposition com.2(9)).

9. $\varphi \equiv \Box\psi$: First suppose that $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$. By **??**, for every Δ' such that $R^\Sigma\Delta\Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By inductive hypothesis, for every Δ' such that $R^\Sigma\Delta\Delta'$, $\psi \in \Delta'$. By definition of R^Σ, for every Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$, $\psi \in \Delta'$. By Proposition com.8, $\Box\psi \in \Delta$.

   Now assume $\Box\psi \in \Delta$. Let $\Delta' \in W^\Sigma$ be such that $R^\Sigma\Delta\Delta'$, i.e., $\Box^{-1}\Delta \subseteq \Delta'$. Since $\Box\psi \in \Delta$, $\psi \in \Box^{-1}\Delta$. Consequently, $\psi \in \Delta'$. By inductive hypothesis, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. Since Δ' is arbitrary with $R^\Sigma\Delta\Delta'$, for all $\Delta' \in W^\Sigma$ such that $R^\Sigma\Delta\Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By **??**, $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$.

10. $\varphi \equiv \Diamond\psi$: First suppose that $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\psi$. By **??**, for some Δ' such that $R^\Sigma\Delta\Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By inductive hypothesis, for some Δ' such that $R^\Sigma\Delta\Delta'$, $\psi \in \Delta'$. By definition of R^Σ, for some Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$, $\psi \in \Delta'$. By Proposition com.10, for some Δ' such that $\Diamond\Delta' \subseteq \Delta$, $\psi \in \Delta'$. Since $\psi \in \Delta'$, $\Diamond\psi \in \Diamond\Delta'$, so $\Diamond\psi \in \Delta$.

    Now assume $\Diamond\psi \in \Delta$. By Proposition com.10, there is a complete Σ-consistent $\Delta' \in W^\Sigma$ such that $\Diamond\Delta' \subseteq \Delta$ and $\psi \in \Delta'$. By Lemma com.9, there is a $\Delta' \in W^\Sigma$ such that $\Box^{-1}\Delta \subseteq \Delta'$, and $\psi \in \Delta'$. By definition of R^Σ, $R^\Sigma\Delta\Delta'$, so there is a $\Delta' \in W^\Sigma$ such that $R^\Sigma\Delta\Delta'$ and $\psi \in \Delta'$. By **??**, $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\psi$. □

Problem com.3. Complete the proof of Proposition com.12.

com.7 Determination and Completeness for K

We are now prepared to use the canonical model to establish completeness. Completeness follows from the fact that the formulas true in the canonical model for Σ are exactly the Σ-derivable ones. Models with this property are said to *determine* Σ. nml:com:cmk:sec

Definition com.13. A model $\mathfrak{M}$ *determines* a normal modal logic Σ precisely when $\mathfrak{M} \Vdash \varphi$ if and only if $\Sigma \vdash \varphi$, for all formulas φ.

Theorem com.14 (Determination). $\mathfrak{M}^\Sigma \Vdash \varphi$ *if and only if* $\Sigma \vdash \varphi$. *nml:com:cmk:thm:determination*

Proof. If $\mathfrak{M}^{\Sigma} \Vdash \varphi$, then for every complete Σ-consistent Δ, we have $\mathfrak{M}^{\Sigma}, \Delta \Vdash \varphi$. Hence, by the Truth Lemma, $\varphi \in \Delta$ for every complete Σ-consistent Δ, whence by Corollary com.4 (with $\Gamma = \emptyset$), $\Sigma \vdash \varphi$.

Conversely, if $\Sigma \vdash \varphi$ then by Proposition com.2(1), every complete Σ-consistent Δ contains φ, and hence by the Truth Lemma, $\mathfrak{M}^{\Sigma}, \Delta \Vdash \varphi$ for every $\Delta \in W^{\Sigma}$, i.e., $\mathfrak{M}^{\Sigma} \Vdash \varphi$. □

Since the canonical model for **K** determines **K**, we immediately have completeness of **K** as a corollary:

nml:com:cmk: cor:Kcomplete

Corollary com.15. *The basic modal logic* **K** *is complete with respect to the class of all models, i.e., if* $\vDash \varphi$ *then* $\mathbf{K} \vdash \varphi$.

Proof. Contrapositively, if $\mathbf{K} \nvdash \varphi$ then by Determination $\mathfrak{M}^{\mathbf{K}} \nVdash \varphi$ and hence φ is not valid. □

For the general case of completeness of a system Σ with respect to a class of models, e.g., of **KTB4** with respect to the class of reflexive, symmetric, transitive models, determination alone is not enough. We must also show that the canonical model for the system Σ is a member of the class, which does not follow obviously from the canonical model construction—nor is it always true!

com.8 Frame Completeness

nml:com:fra: sec

The completeness theorem for **K** can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

nml:com:fra: thm:completeframeprops

Theorem com.16. *If a normal modal logic* Σ *contains one of the formulas on the left-hand side of Table com.1, then the canonical model for* Σ *has the corresponding property on the right-hand side.*

If Σ contains . . .	*. . . the canonical model for Σ is:*
D: $\Box\varphi \to \Diamond\varphi$	serial;
T: $\Box\varphi \to \varphi$	reflexive;
B: $\varphi \to \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \to \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \to \Box\Diamond\varphi$	euclidean.

Table com.1: Basic correspondence facts.

nml:com:fra: tab:correspondencetable

Proof. We take each of these up in turn.

Suppose Σ contains D, and let $\Delta \in W^{\Sigma}$; we need to show that there is a Δ' such that $R^{\Sigma}\Delta\Delta'$. It suffices to show that $\Box^{-1}\Delta$ is Σ-consistent, for then by Lindenbaum's Lemma, there is a complete Σ-consistent set $\Delta' \supseteq \Box^{-1}\Delta$, and by definition of R^{Σ} we have $R^{\Sigma}\Delta\Delta'$. So, suppose for contradiction that $\Box^{-1}\Delta$ is *not* Σ-consistent, i.e., $\Box^{-1}\Delta \vdash_{\Sigma} \bot$. By Lemma com.7, $\Delta \vdash_{\Sigma} \Box\bot$, and since Σ contains D, also $\Delta \vdash_\Sigma \Diamond\bot$. But Σ is normal, so $\Sigma \vdash \neg\Diamond\bot$ (**??**), whence also $\Delta \vdash_\Sigma \neg\Diamond\bot$, against the consistency of Δ.

Now suppose Σ contains T, and let $\Delta \in W^\Sigma$. We want to show $R^\Sigma\Delta\Delta$, i.e., $\Box^{-1}\Delta \subseteq \Delta$. But if $\Box\varphi \in \Delta$ then by T also $\varphi \in \Delta$, as desired.

Now suppose Σ contains B, and suppose $R^\Sigma\Delta\Delta'$ for Δ, $\Delta' \in W^\Sigma$. We need to show that $R^\Sigma\Delta'\Delta$, i.e., $\Box^{-1}\Delta' \subseteq \Delta$. By Lemma com.9, this is equivalent to $\Diamond\Delta \subseteq \Delta'$. So suppose $\varphi \in \Delta$. By B, also $\Box\Diamond\varphi \in \Delta$. By the hypothesis that $R^\Sigma\Delta\Delta'$, we have that $\Box^{-1}\Delta \subseteq \Delta'$, and hence $\Diamond\varphi \in \Delta'$, as required.

Now suppose Σ contains 4, and suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_2\Delta_3$. We need to show $R^\Sigma\Delta_1\Delta_3$. From the hypothesis we have both $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and $\Box^{-1}\Delta_2 \subseteq \Delta_3$. In order to show $R^\Sigma\Delta_1\Delta_3$ it suffices to show $\Box^{-1}\Delta_1 \subseteq \Delta_3$. So let $\psi \in \Box^{-1}\Delta_1$, i.e., $\Box\psi \in \Delta_1$. By 4, also $\Box\Box\psi \in \Delta_1$ and by hypothesis we get, first, that $\Box\psi \in \Delta_2$ and, second, that $\psi \in \Delta_3$, as desired.

Now suppose Σ contains 5, suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$. We need to show $R^\Sigma\Delta_2\Delta_3$. The first hypothesis gives $\Box^{-1}\Delta_1 \subseteq \Delta_2$, and the second hypothesis is equivalent to $\Diamond\Delta_3 \subseteq \Delta_2$, by Lemma com.9. To show $R^\Sigma\Delta_2\Delta_3$, by Lemma com.9, it suffices to show $\Diamond\Delta_3 \subseteq \Delta_2$. So let $\Diamond\varphi \in \Diamond\Delta_3$, i.e., $\varphi \in \Delta_3$. By the second hypothesis $\Diamond\varphi \in \Delta_1$ and by 5, $\Box\Diamond\varphi \in \Delta_1$ as well. But now the first hypothesis gives $\Diamond\varphi \in \Delta_2$, as desired. □

As a corollary we obtain completeness results for a number of systems. For instance, we know that **S5** = **KT5** = **KTB4** is complete with respect to the class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

Theorem com.17. *Let $\mathcal{C}_\mathrm{D}$, $\mathcal{C}_\mathrm{T}$, $\mathcal{C}_\mathrm{B}$, $\mathcal{C}_4$, and $\mathcal{C}_5$ be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1, \dots, \varphi_n$ among* D, T, B, 4, *and* 5, *the system $\mathbf{K}\varphi_1 \dots \varphi_n$ is determined by the class of models $\mathcal{C} = \mathcal{C}_{\varphi_1} \cap \dots \cap \mathcal{C}_{\varphi_n}$.* *nml:com:fra:thm:generaldet*

Proposition com.18. *Let Σ be a normal modal logic; then:*

1. *If Σ contains the schema $\Diamond\varphi \to \Box\varphi$ then the canonical model for Σ is partially functional.* *nml:com:fra:prop:anotherfive-a*

2. *If Σ contains the schema $\Diamond\varphi \leftrightarrow \Box\varphi$ then the canonical model for Σ is functional.*

3. *If Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ then the canonical model for Σ is weakly dense.*

(see **??** *for definitions of these frame properties).*

Proof. 1. Suppose that Σ contains the schema $\Diamond\varphi \to \Box\varphi$, to show that R^Σ is partially functional we need to prove that for any Δ_1, Δ_2, $\Delta_3 \in W^\Sigma$, if $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$ then $\Delta_2 = \Delta_3$. Since $R^\Sigma\Delta_1\Delta_2$ we have $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and since $R^\Sigma\Delta_1\Delta_3$ also $\Box^{-1}\Delta_1 \subseteq \Delta_3$. The identity $\Delta_2 = \Delta_3$ will follow if we can establish the two inclusions $\Delta_2 \subseteq \Delta_3$ and $\Delta_3 \subseteq$

Δ_2. For the first inclusion, let $\varphi \in \Delta_2$; then $\Diamond\varphi \in \Delta_1$, and by the schema and deductive closure of Δ_1 also $\Box\varphi \in \Delta_1$, whence by the hypothesis that $R^\Sigma \Delta_1 \Delta_3$, $\varphi \in \Delta_3$. The second inclusion is similar.

2. This follows immediately from part (1) and the seriality proof in Theorem com.16.

3. Suppose Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ and to show that R^Σ is weakly dense, let $R^\Sigma \Delta_1 \Delta_2$. We need to show that there is a complete Σ-consistent set Δ_3 such that $R^\Sigma \Delta_1 \Delta_3$ and $R^\Sigma \Delta_3 \Delta_2$. Let:

$$\Gamma = \Box^{-1}\Delta_1 \cup \Diamond\Delta_2.$$

It suffices to show that Γ is Σ-consistent, for then by Lindenbaum's Lemma it can be extended to a complete Σ-consistent set Δ_3 such that $\Box^{-1}\Delta_1 \subseteq \Delta_3$ and $\Diamond\Delta_2 \subseteq \Delta_3$, i.e., $R^\Sigma \Delta_1 \Delta_3$ and $R^\Sigma \Delta_3 \Delta_2$ (by Lemma com.9).

Suppose for contradiction that Γ is not consistent. Then there are formulas $\Box\varphi_1, \dots, \Box\varphi_n \in \Delta_1$ and $\psi_1, \dots, \psi_m \in \Delta_2$ such that

$$\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_\Sigma \bot.$$

Since $\Diamond(\psi_1 \land \dots \land \psi_m) \to (\Diamond\psi_1 \land \dots \land \Diamond\psi_m)$ is derivable in every normal modal logic, we argue as follows, contradicting the consistency of Δ_2:

$$\begin{aligned}
\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m &\vdash_\Sigma \bot \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma (\Diamond\psi_1 \land \dots \land \Diamond\psi_m) \to \bot \\
&\quad \text{by the deduction theorem} \\
&\quad \textbf{????}, \text{ and TAUT} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \Diamond(\psi_1 \land \dots \land \psi_m) \to \bot \\
&\quad \text{since } \Sigma \text{ is normal} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \neg\Diamond(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by PL} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m) \\
&\quad \Box\neg \text{ for } \neg\Diamond \\
\Box\varphi_1, \dots, \Box\varphi_n &\vdash_\Sigma \Box\Box\neg(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by Lemma com.6} \\
\Box\varphi_1, \dots, \Box\varphi_n &\vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by schema } \Box\Box\varphi \to \Box\varphi \\
\Delta_1 &\vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by monotonicity, } \textbf{????} \\
\Box\neg(\psi_1 \land \dots \land \psi_m) &\in \Delta_1 \\
&\quad \text{by deductive closure;} \\
\neg(\psi_1 \land \dots \land \psi_m) &\in \Delta_2 \\
&\quad \text{since } R^\Sigma \Delta_1 \Delta_2.
\end{aligned}$$

□

On the strength of these examples, one might think that every system Σ of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of Σ is valid. Unfortunately, there are many systems that are not complete in this sense.

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Bibliography