Determination and Completeness for $K$

We are now prepared to use the canonical model to establish completeness. Completeness follows from the fact that the formulas true in the canonical model for $\Sigma$ are exactly the $\Sigma$-derivable ones. Models with this property are said to determine $\Sigma$.

**Definition com.1.** A model $\mathfrak{M}$ determines a normal modal logic $\Sigma$ precisely when $\mathfrak{M} \models \varphi$ if and only if $\Sigma \vdash \varphi$, for all formulas $\varphi$.

**Theorem com.2 (Determination).** $\mathfrak{M}^\Sigma \models \varphi$ if and only if $\Sigma \vdash \varphi$.

*Proof.* If $\mathfrak{M}^\Sigma \models \varphi$, then for every complete $\Sigma$-consistent $\Delta$, we have $\mathfrak{M}^\Sigma, \Delta \models \varphi$. Hence, by the Truth Lemma, $\varphi \in \Delta$ for every complete $\Sigma$-consistent $\Delta$, whence by ?? (with $\Gamma = \emptyset$), $\Sigma \vdash \varphi$.

Conversely, if $\Sigma \vdash \varphi$ then by ??, every complete $\Sigma$-consistent $\Delta$ contains $\varphi$, and hence by the Truth Lemma, $\mathfrak{M}^\Sigma, \Delta \models \varphi$ for every $\Delta \in W^\Sigma$, i.e., $\mathfrak{M}^\Sigma \models \varphi$. $\Box$

Since the canonical model for $K$ determines $K$, we immediately have completeness of $K$ as a corollary:

**Corollary com.3.** The basic modal logic $K$ is complete with respect to the class of all models, i.e., if $\models \varphi$ then $K \vdash \varphi$.

*Proof.* Contrapositively, if $K \not\vdash \varphi$ then by Determination $\mathfrak{M}^K \not\models \varphi$ and hence $\varphi$ is not valid. $\Box$

For the general case of completeness of a system $\Sigma$ with respect to a class of models, e.g., of $KTB4$ with respect to the class of reflexive, symmetric, transitive models, determination alone is not enough. We must also show that the canonical model for the system $\Sigma$ is a member of the class, which does not follow obviously from the canonical model construction—nor is it always true!

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**Bibliography**