com.1 Determination and Completeness for K

mod:com:cmk: sec

We are now prepared to use the canonical model to establish completeness. Completeness follows from the fact that the formulas true in the canonical for Σ are exactly the Σ-derivable ones. Models with this property are said to *determine* Σ.

Definition com.1. A model $\mathfrak{M}$ *determines* a normal modal logic Σ precisely when $\mathfrak{M} \Vdash \varphi$ if and only if $\Sigma \vdash \varphi$, for all formulas φ.

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Theorem com.2 (Determination). *$\mathfrak{M}^\Sigma \Vdash \varphi$ if and only if $\Sigma \vdash \varphi$.*

Proof. If $\mathfrak{M}^\Sigma \Vdash \varphi$, then for every complete Σ-consistent Δ, we have $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$. Hence, by the Truth Lemma, $\varphi \in \Delta$ for every complete Σ-consistent Δ, whence by **??** (with $\Gamma = \emptyset$), $\Sigma \vdash \varphi$.

Conversely, if $\Sigma \vdash \varphi$ then by **????**, every complete Σ-consistent Δ contains φ, and hence by the Truth Lemma, $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ for every $\Delta \in W^\Sigma$, i.e., $\mathfrak{M}^\Sigma \Vdash \varphi$. □

Since the canonical model for **K** determines **K**, we immediately have completeness of **K** as a corollary:

mod:com:cmk: cor:Kcomplete

Corollary com.3. *The basic modal logic* **K** *is complete with respect to the class of all models, i.e., if $\vDash \varphi$ then* $\mathbf{K} \vdash \varphi$.

Proof. Contrapositively, if $\mathbf{K} \nvdash \varphi$ then by Determination $\mathfrak{M}^{\mathbf{K}} \nVdash \varphi$ and hence φ is not valid. □

For the general case of completeness of a system Σ with respect to a class of models, e.g., of **KTB4** with respect to the class of reflexive, symmetric, transitive models, determination alone is not enough. We must also show that the canonical model for the system Σ is a member of the class, which does not follow obviously from the canoncial model construction—nor is it always true!

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Bibliography