

## com.1 Determination and Completeness for $\mathbf{K}$

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sec

We are now prepared to use the canonical model to establish completeness. Completeness follows from the fact that the formulas true in the canonical for  $\Sigma$  are exactly the  $\Sigma$ -derivable ones. Models with this property are said to *determine*  $\Sigma$ .

**Definition com.1.** A model  $\mathfrak{M}$  *determines* a normal modal logic  $\Sigma$  precisely when  $\mathfrak{M} \Vdash \varphi$  if and only if  $\Sigma \vdash \varphi$ , for all formulas  $\varphi$ .

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thm:determination

**Theorem com.2** (Determination).  $\mathfrak{M}^\Sigma \Vdash \varphi$  if and only if  $\Sigma \vdash \varphi$ .

*Proof.* If  $\mathfrak{M}^\Sigma \Vdash \varphi$ , then for every complete  $\Sigma$ -consistent  $\Delta$ , we have  $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ . Hence, by the Truth Lemma,  $\varphi \in \Delta$  for every complete  $\Sigma$ -consistent  $\Delta$ , whence by ?? (with  $\Gamma = \emptyset$ ),  $\Sigma \vdash \varphi$ .

Conversely, if  $\Sigma \vdash \varphi$  then by ?????, every complete  $\Sigma$ -consistent  $\Delta$  contains  $\varphi$ , and hence by the Truth Lemma,  $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$  for every  $\Delta \in W^\Sigma$ , i.e.,  $\mathfrak{M}^\Sigma \Vdash \varphi$ .  $\square$

Since the canonical model for  $\mathbf{K}$  determines  $\mathbf{K}$ , we immediately have completeness of  $\mathbf{K}$  as a corollary:

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cor:Kcomplete

**Corollary com.3.** *The basic modal logic  $\mathbf{K}$  is complete with respect to the class of all models, i.e., if  $\vDash \varphi$  then  $\mathbf{K} \vdash \varphi$ .*

*Proof.* Contrapositively, if  $\mathbf{K} \not\vdash \varphi$  then by Determination  $\mathfrak{M}^\mathbf{K} \not\vdash \varphi$  and hence  $\varphi$  is not valid.  $\square$

For the general case of completeness of a system  $\Sigma$  with respect to a class of models, e.g., of  $\mathbf{KTB4}$  with respect to the class of reflexive, symmetric, transitive models, determination alone is not enough. We must also show that the canonical model for the system  $\Sigma$  is a member of the class, which does not follow obviously from the canonical model construction—nor is it always true!

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## Bibliography