com.1 Complete Σ-Consistent Sets

mod:com:ccs: sec

Suppose Σ is a set of modal formulas—think of them as the axioms or defining principles of a normal modal logic. A set Γ is Σ-consistent iff $\Gamma \nvdash_\Sigma \bot$, i.e., if there is no derivation of $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \bot) \dots)$ from Σ, where each $\varphi_i \in \Gamma$. We will construct a "canonical" model in which each world is taken to be a special kind of Σ-consistent set: one which is not just Σ-consistent, but maximally so, in the sense that it settles the truth value of every modal formula: for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$:

Definition com.1. A set Γ is *complete Σ-consistent* if and only if it is Σ-consistent and for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Complete Σ-consistent sets Γ have a number of useful properties. For one, they are deductively closed, i.e., if $\Gamma \vdash_\Sigma \varphi$ then $\varphi \in \Gamma$. This means in particular that every instance of a formula $\varphi \in \Sigma$ is also $\in \Gamma$. Moreover, membership in Γ mirrors the truth conditions for the propositional connectives. This will be important when we define the "canonical model."

mod:com:ccs: prop:ccs-properties

Proposition com.2. *Suppose Γ is complete Σ-consistent. Then:*

mod:com:ccs: prop:ccs-closed

1. *Γ is deductively closed in Σ.*

mod:com:ccs: prop:ccs-sigma

2. *$\Sigma \subseteq \Gamma$.*

mod:com:ccs: prop:ccs-lfalse

3. *$\bot \notin \Gamma$*

mod:com:ccs: prop:ccs-ltrue

4. *$\top \in \Gamma$*

mod:com:ccs: prop:ccs-lnot

5. *$\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.*

mod:com:ccs: prop:ccs-land

6. *$\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$*

mod:com:ccs: prop:ccs-lor

7. *$\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$*

mod:com:ccs: prop:ccs-lif

8. *$\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$*

mod:com:ccs: prop:ccs-liff

9. *$\varphi \leftrightarrow \psi \in \Gamma$ iff either $\varphi \in \Gamma$ and $\psi \in \Gamma$, or $\varphi \notin \Gamma$ and $\psi \notin \Gamma$*

Proof.

1. Suppose $\Gamma \vdash_\Sigma \varphi$ but $\varphi \notin \Gamma$. Then since Γ is complete Σ-consistent, $\neg\varphi \in \Gamma$. This would make Γ inconsistent, since $\varphi, \neg\varphi \vdash_\Sigma \bot$.
2. If $\varphi \in \Sigma$ then $\Gamma \vdash_\Sigma \varphi$, and $\varphi \in \Gamma$ by deductive closure, i.e., case (1).
3. If $\bot \in \Gamma$, then $\Gamma \vdash_\Sigma \bot$, so Γ would be Σ-inconsistent.
4. $\Gamma \vdash_\Sigma \top$, so $\top \in \Gamma$ by deductive closure, i.e., case (1).
5. If $\neg\varphi \in \Gamma$, then by consistency $\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then $\varphi \in \Gamma$ since Γ is complete Σ-consistent.

6. Suppose $\varphi \wedge \psi \in \Gamma$. Since $(\varphi \wedge \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure, i.e., case (1). Similiarly for $\psi \in \Gamma$. On the other hand, suppose both $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then deductive closure implies $(\varphi \wedge \psi) \in \Gamma$, since $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$ is a tautological instance.

7. Suppose $\varphi \vee \psi \in \Gamma$, and $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is complete Σ-consistent, $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. Then $\neg(\varphi \vee \psi) \in \Gamma$ since $\neg\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \vee \psi))$ is a tautological instance. This would mean that Γ is Σ-inconsistent, a contradiction.

8. Suppose $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash_{\Sigma} \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \rightarrow \psi \notin \Gamma$ then since Γ is complete Σ-consistent, $\neg(\varphi \rightarrow \psi) \in \Gamma$. Since $\neg(\varphi \rightarrow \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure. Since $\neg(\varphi \rightarrow \psi) \rightarrow \neg\psi$ is a tautological instance, $\neg\psi \in \Gamma$. Then $\psi \notin \Gamma$ since Γ is Σ-consistent.

9. Suppose $\varphi \leftrightarrow \psi \in \Gamma$. If $\varphi \in \Gamma$, then $\psi \in \Gamma$, since $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$ is a tautological instance. Similarly, if $\psi \in \Gamma$, then $\varphi \in \Gamma$. So either both $\varphi \in \Gamma$ and $\psi \in \Gamma$, or neither $\varphi \in \Gamma$ nor $\psi \in \Gamma$.

   Conversely, suppose $\varphi \rightarrow \psi \notin \Gamma$. Since Γ is complete Σ-consistent, $\neg(\varphi \leftrightarrow \psi) \in \Gamma$. Since $\neg(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \neg\psi)$ is a tautological instance, if $\varphi \in \Gamma$ then $\neg\psi \in \Gamma$, and since Γ is Σ-consistent, $\psi \notin \Gamma$. Similarly, if $\psi \in \Gamma$ then $\varphi \notin \Gamma$. So neither $\varphi \in \Gamma$ and $\psi \in \Gamma$, nor $\varphi \notin \Gamma$ and $\psi \notin \Gamma$.

□

Problem com.1. Complete the proof of Proposition com.2.

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Bibliography