com.1  Complete Σ-Consistent Sets

Suppose Σ is a set of modal formulas—think of them as the axioms or defining principles of a normal modal logic. A set Γ is Σ-consistent iff Γ ⊬ Σ ⊥, i.e., if there is no derivation of φ₁ → (φ₂ → ··· (φₙ → ⊥) ···) from Σ, where each φᵢ ∈ Γ. We will construct a “canonical” model in which each world is taken to be a special kind of Σ-consistent set: one which is not just Σ-consistent, but maximally so, in the sense that it settles the truth value of every modal formula: for every φ, either φ ∈ Γ or ¬φ ∈ Γ:

**Definition com.1.** A set Γ is complete Σ-consistent if and only if it is Σ-consistent and for every φ, either φ ∈ Γ or ¬φ ∈ Γ.

Complete Σ-consistent sets Γ have a number of useful properties. For one, they are deductively closed, i.e., if Γ ⊬ Σ φ then φ /∈ Γ. This means in particular that every instance of a formula φ ∈ Σ is also ∈ Γ. Moreover, membership in Γ mirrors the truth conditions for the propositional connectives. This will be important when we define the “canonical model.”

**Proposition com.2.** Suppose Γ is complete Σ-consistent. Then:

1. Γ is deductively closed in Σ.
2. Σ ⊆ Γ.
3. ⊥ ∉ Γ.
4. ⊤ ∈ Γ.
5. ¬φ ∈ Γ if and only if φ /∈ Γ.
6. φ ∧ ψ ∈ Γ iff φ ∈ Γ and ψ ∈ Γ.
7. φ ∨ ψ ∈ Γ iff φ ∈ Γ or ψ ∈ Γ.
8. φ → ψ ∈ Γ iff φ /∈ Γ or ψ ∈ Γ.
9. φ ↔ ψ ∈ Γ iff either φ ∈ Γ and ψ ∈ Γ, or φ /∈ Γ and ψ /∈ Γ.

**Proof.**

1. Suppose Γ ⊬ Σ φ but φ /∈ Γ. Then since Γ is complete Σ-consistent, ¬φ ∈ Γ. This would make Γ inconsistent, since φ, ¬φ ⊬ Σ ⊥.

2. If φ ∈ Σ then Γ ⊬ Σ φ, and φ ∈ Γ by deductive closure, i.e., case (1).

3. If ⊥ ∈ Γ, then Γ ⊬ Σ ⊥, so Γ would be Σ-inconsistent.

4. Γ ⊬ Σ ⊤, so ⊤ ∈ Γ by deductive closure, i.e., case (1).

5. If ¬φ ∈ Γ, then by consistency φ /∈ Γ; and if φ /∈ Γ then φ ∈ Γ since Γ is complete Σ-consistent.
6. Suppose $\varphi \land \psi \in \Gamma$. Since $(\varphi \land \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure, i.e., case (1). Similarly for $\psi \in \Gamma$. On the other hand, suppose both $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then deductive closure implies $(\varphi \land \psi) \in \Gamma$, since $\varphi \rightarrow (\psi \rightarrow (\varphi \land \psi))$ is a tautological instance.

7. Suppose $\varphi \lor \psi \in \Gamma$, and $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since $\Gamma$ is complete $\Sigma$-consistent, $\neg \varphi \in \Gamma$ and $\neg \psi \in \Gamma$. Then $\neg(\varphi \lor \psi) \in \Gamma$ since $\neg \varphi \rightarrow (\neg \psi \rightarrow \neg(\varphi \lor \psi))$ is a tautological instance. This would mean that $\Gamma$ is $\Sigma$-inconsistent, a contradiction.

8. Suppose $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash \Sigma \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \rightarrow \psi \notin \Gamma$ then since $\Gamma$ is complete $\Sigma$-consistent, $\neg(\varphi \rightarrow \psi) \in \Gamma$. Since $\neg(\varphi \rightarrow \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure. Since $\neg(\varphi \rightarrow \psi) \rightarrow \neg \psi$ is a tautological instance, $\neg \psi \in \Gamma$. Then $\psi \notin \Gamma$ since $\Gamma$ is $\Sigma$-consistent.

9. Suppose $\varphi \leftrightarrow \psi \in \Gamma$. If $\varphi \in \Gamma$, then $\psi \in \Gamma$, since $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$ is a tautological instance. Similarly, if $\psi \in \Gamma$, then $\varphi \in \Gamma$. So either both $\varphi \in \Gamma$ and $\psi \in \Gamma$, or neither $\varphi \in \Gamma$ nor $\psi \in \Gamma$.

Conversely, suppose $\varphi \rightarrow \psi \notin \Gamma$. Since $\Gamma$ is complete $\Sigma$-consistent, $\neg(\varphi \leftrightarrow \psi) \in \Gamma$. Since $\neg(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \neg \psi)$ is a tautological instance, if $\varphi \in \Gamma$ then $\neg \psi \in \Gamma$, and since $\Gamma$ is $\Sigma$-consistent, $\psi \notin \Gamma$. Similarly, if $\psi \in \Gamma$ then $\varphi \notin \Gamma$. So neither $\varphi \in \Gamma$ and $\psi \in \Gamma$, nor $\varphi \notin \Gamma$ and $\psi \notin \Gamma$.

**Problem com.1.** Complete the proof of Proposition com.2.

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