

com.1 Complete Σ -Consistent Sets

mod:com:ccs:
sec

Suppose Σ is a set of modal **formulas**—think of them as the axioms or defining principles of a normal modal logic. A set Γ is Σ -consistent iff $\Gamma \not\vdash_{\Sigma} \perp$, i.e., if there is no **derivation** of $\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_n \rightarrow \perp) \dots)$ from Σ , where each $\varphi_i \in \Gamma$. We will construct a “canonical” model in which each world is taken to be a special kind of Σ -consistent set: one which is not just Σ -consistent, but maximally so, in the sense that it settles the truth value of every modal **formula**: for every φ , either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$:

Definition com.1. A set Γ is *complete Σ -consistent* if and only if it is Σ -consistent and for every φ , either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Complete Σ -consistent sets Γ have a number of useful properties. For one, they are deductively closed, i.e., if $\Gamma \vdash_{\Sigma} \varphi$ then $\varphi \in \Gamma$. This means in particular that every instance of a **formula** $\varphi \in \Sigma$ is also $\in \Gamma$. Moreover, membership in Γ mirrors the truth conditions for the propositional connectives. This will be important when we define the “canonical model.”

Proposition com.2. *Suppose Γ is complete Σ -consistent. Then:*

mod:com:ccs:
prop:ccs-properties

1. Γ is deductively closed in Σ .

mod:com:ccs:
prop:ccs-closed

2. $\Sigma \subseteq \Gamma$.

mod:com:ccs:
prop:ccs-sigma

3. $\perp \notin \Gamma$

mod:com:ccs:
prop:ccs-lfalse

4. $\top \in \Gamma$

mod:com:ccs:
prop:ccs-ltrue

5. $\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.

mod:com:ccs:
prop:ccs-lnot

6. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$

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prop:ccs-land

7. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$

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prop:ccs-lor

8. $\varphi \rightarrow \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$

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prop:ccs-lif

9. $\varphi \leftrightarrow \psi \in \Gamma$ iff either $\varphi \in \Gamma$ and $\psi \in \Gamma$, or $\varphi \notin \Gamma$ and $\psi \notin \Gamma$

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prop:ccs-liff

Proof. 1. Suppose $\Gamma \vdash_{\Sigma} \varphi$ but $\varphi \notin \Gamma$. Then since Γ is complete Σ -consistent, $\neg\varphi \in \Gamma$. This would make Γ inconsistent, since $\varphi, \neg\varphi \vdash_{\Sigma} \perp$.

2. If $\varphi \in \Sigma$ then $\Gamma \vdash_{\Sigma} \varphi$, and $\varphi \in \Gamma$ by deductive closure, i.e., case (1).

3. If $\perp \in \Gamma$, then $\Gamma \vdash_{\Sigma} \perp$, so Γ would be Σ -inconsistent.

4. $\Gamma \vdash_{\Sigma} \top$, so $\top \in \Gamma$ by deductive closure, i.e., case (1).

5. If $\neg\varphi \in \Gamma$, then by consistency $\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then $\varphi \in \Gamma$ since Γ is complete Σ -consistent.

6. Suppose $\varphi \wedge \psi \in \Gamma$. Since $(\varphi \wedge \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure, i.e., case (1). Similarly for $\psi \in \Gamma$. On the other hand, suppose both $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then deductive closure implies $(\varphi \wedge \psi) \in \Gamma$, since $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$ is a tautological instance.
7. Suppose $\varphi \vee \psi \in \Gamma$, and $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is complete Σ -consistent, $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. Then $\neg(\varphi \vee \psi) \in \Gamma$ since $\neg\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \vee \psi))$ is a tautological instance. This would mean that Γ is Σ -inconsistent, a contradiction.
8. Suppose $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash_{\Sigma} \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \rightarrow \psi \notin \Gamma$ then since Γ is complete Σ -consistent, $\neg(\varphi \rightarrow \psi) \in \Gamma$. Since $\neg(\varphi \rightarrow \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure. Since $\neg(\varphi \rightarrow \psi) \rightarrow \neg\psi$ is a tautological instance, $\neg\psi \in \Gamma$. Then $\psi \notin \Gamma$ since Γ is Σ -consistent.
9. Suppose $\varphi \leftrightarrow \psi \in \Gamma$. If $\varphi \in \Gamma$, then $\psi \in \Gamma$, since $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$ is a tautological instance. Similarly, if $\psi \in \Gamma$, then $\varphi \in \Gamma$. So either both $\varphi \in \Gamma$ and $\psi \in \Gamma$, or neither $\varphi \in \Gamma$ nor $\psi \in \Gamma$.
Conversely, suppose $\varphi \rightarrow \psi \notin \Gamma$. Since Γ is complete Σ -consistent, $\neg(\varphi \leftrightarrow \psi) \in \Gamma$. Since $\neg(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \neg\psi)$ is a tautological instance, if $\varphi \in \Gamma$ then $\neg\psi \in \Gamma$, and since Γ is Σ -consistent, $\psi \notin \Gamma$. Similarly, if $\psi \in \Gamma$ then $\varphi \notin \Gamma$. So neither $\varphi \in \Gamma$ and $\psi \in \Gamma$, nor $\varphi \notin \Gamma$ and $\psi \notin \Gamma$.

□

Problem com.1. Complete the proof of [Proposition com.2](#).

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Bibliography