

## com.1 Complete $\Sigma$ -Consistent Sets

mod:com:ccs:  
sec

Suppose  $\Sigma$  is a set of modal **formulas**—think of them as the axioms or defining principles of a normal modal logic. A set  $\Gamma$  is  $\Sigma$ -consistent iff  $\Gamma \not\vdash_{\Sigma} \perp$ , i.e., if there is no **derivation** of  $\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_n \rightarrow \perp) \dots)$  from  $\Sigma$ , where each  $\varphi_i \in \Gamma$ . We will construct a “canonical” model in which each world is taken to be a special kind of  $\Sigma$ -consistent set: one which is not just  $\Sigma$ -consistent, but maximally so, in the sense that it settles the truth value of every modal **formula**: for every  $\varphi$ , either  $\varphi \in \Gamma$  or  $\neg\varphi \in \Gamma$ :

**Definition com.1.** A set  $\Gamma$  is *complete  $\Sigma$ -consistent* if and only if it is  $\Sigma$ -consistent and for every  $\varphi$ , either  $\varphi \in \Gamma$  or  $\neg\varphi \in \Gamma$ .

Complete  $\Sigma$ -consistent sets  $\Gamma$  have a number of useful properties. For one, they are deductively closed, i.e., if  $\Gamma \vdash_{\Sigma} \varphi$  then  $\varphi \in \Gamma$ . This means in particular that every instance of a **formula**  $\varphi \in \Sigma$  is also  $\in \Gamma$ . Moreover, membership in  $\Gamma$  mirrors the truth conditions for the propositional connectives. This will be important when we define the “canonical model.”

**Proposition com.2.** *Suppose  $\Gamma$  is complete  $\Sigma$ -consistent. Then:*

mod:com:ccs:  
prop:ccs-properties

1.  $\Gamma$  is deductively closed in  $\Sigma$ .

mod:com:ccs:  
prop:ccs-closed

2.  $\Sigma \subseteq \Gamma$ .

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prop:ccs-sigma

3.  $\perp \notin \Gamma$

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prop:ccs-lfalse

4.  $\top \in \Gamma$

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prop:ccs-ltrue

5.  $\neg\varphi \in \Gamma$  if and only if  $\varphi \notin \Gamma$ .

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prop:ccs-lnot

6.  $\varphi \wedge \psi \in \Gamma$  iff  $\varphi \in \Gamma$  and  $\psi \in \Gamma$

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prop:ccs-land

7.  $\varphi \vee \psi \in \Gamma$  iff  $\varphi \in \Gamma$  or  $\psi \in \Gamma$

mod:com:ccs:  
prop:ccs-lor

8.  $\varphi \rightarrow \psi \in \Gamma$  iff  $\varphi \notin \Gamma$  or  $\psi \in \Gamma$

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prop:ccs-lif

9.  $\varphi \leftrightarrow \psi \in \Gamma$  iff either  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ , or  $\varphi \notin \Gamma$  and  $\psi \notin \Gamma$

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prop:ccs-liff

*Proof.* 1. Suppose  $\Gamma \vdash_{\Sigma} \varphi$  but  $\varphi \notin \Gamma$ . Then since  $\Gamma$  is complete  $\Sigma$ -consistent,  $\neg\varphi \in \Gamma$ . This would make  $\Gamma$  inconsistent, since  $\varphi, \neg\varphi \vdash_{\Sigma} \perp$ .

2. If  $\varphi \in \Sigma$  then  $\Gamma \vdash_{\Sigma} \varphi$ , and  $\varphi \in \Gamma$  by deductive closure, i.e., case (1).

3. If  $\perp \in \Gamma$ , then  $\Gamma \vdash_{\Sigma} \perp$ , so  $\Gamma$  would be  $\Sigma$ -inconsistent.

4.  $\Gamma \vdash_{\Sigma} \top$ , so  $\top \in \Gamma$  by deductive closure, i.e., case (1).

5. If  $\neg\varphi \in \Gamma$ , then by consistency  $\varphi \notin \Gamma$ ; and if  $\varphi \notin \Gamma$  then  $\varphi \in \Gamma$  since  $\Gamma$  is complete  $\Sigma$ -consistent.

6. Suppose  $\varphi \wedge \psi \in \Gamma$ . Since  $(\varphi \wedge \psi) \rightarrow \varphi$  is a tautological instance,  $\varphi \in \Gamma$  by deductive closure, i.e., case (1). Similarly for  $\psi \in \Gamma$ . On the other hand, suppose both  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ . Then deductive closure implies  $(\varphi \wedge \psi) \in \Gamma$ , since  $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$  is a tautological instance.
7. Suppose  $\varphi \vee \psi \in \Gamma$ , and  $\varphi \notin \Gamma$  and  $\psi \notin \Gamma$ . Since  $\Gamma$  is complete  $\Sigma$ -consistent,  $\neg\varphi \in \Gamma$  and  $\neg\psi \in \Gamma$ . Then  $\neg(\varphi \vee \psi) \in \Gamma$  since  $\neg\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \vee \psi))$  is a tautological instance. This would mean that  $\Gamma$  is  $\Sigma$ -inconsistent, a contradiction.
8. Suppose  $\varphi \rightarrow \psi \in \Gamma$  and  $\varphi \in \Gamma$ ; then  $\Gamma \vdash_{\Sigma} \psi$ , whence  $\psi \in \Gamma$  by deductive closure. Conversely, if  $\varphi \rightarrow \psi \notin \Gamma$  then since  $\Gamma$  is complete  $\Sigma$ -consistent,  $\neg(\varphi \rightarrow \psi) \in \Gamma$ . Since  $\neg(\varphi \rightarrow \psi) \rightarrow \varphi$  is a tautological instance,  $\varphi \in \Gamma$  by deductive closure. Since  $\neg(\varphi \rightarrow \psi) \rightarrow \neg\psi$  is a tautological instance,  $\neg\psi \in \Gamma$ . Then  $\psi \notin \Gamma$  since  $\Gamma$  is  $\Sigma$ -consistent.
9. Suppose  $\varphi \leftrightarrow \psi \in \Gamma$ . If  $\varphi \in \Gamma$ , then  $\psi \in \Gamma$ , since  $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$  is a tautological instance. Similarly, if  $\psi \in \Gamma$ , then  $\varphi \in \Gamma$ . So either both  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ , or neither  $\varphi \in \Gamma$  nor  $\psi \in \Gamma$ .  
Conversely, suppose  $\varphi \rightarrow \psi \notin \Gamma$ . Since  $\Gamma$  is complete  $\Sigma$ -consistent,  $\neg(\varphi \leftrightarrow \psi) \in \Gamma$ . Since  $\neg(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \neg\psi)$  is a tautological instance, if  $\varphi \in \Gamma$  then  $\neg\psi \in \Gamma$ , and since  $\Gamma$  is  $\Sigma$ -consistent,  $\psi \notin \Gamma$ . Similarly, if  $\psi \in \Gamma$  then  $\varphi \notin \Gamma$ . So neither  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ , nor  $\varphi \notin \Gamma$  and  $\psi \notin \Gamma$ .

□

**Problem com.1.** Complete the proof of [Proposition com.2](#).

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## Bibliography