

prf.1 Showing Systems are Distinct

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sec

In ?? we saw how to prove that two systems of modal logic are in fact the same system. ?? allows us to show that two modal systems Σ and Σ' are distinct, by finding a formula φ such that $\Sigma' \vdash \varphi$ that fails in a model of Σ .

Proposition prf.1. $\mathbf{KD} \subsetneq \mathbf{KT}$

Proof. This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show $\mathbf{KD} \subseteq \mathbf{KT}$ we need to see that $\mathbf{KD} \vdash \psi$ implies $\mathbf{KT} \vdash \psi$, which follows from $\mathbf{KT} \vdash \mathbf{D}$, as shown in ????. To show that the inclusion is proper, by Soundness (??), it suffices to exhibit a model of \mathbf{KD} where some instance $\Box\varphi \rightarrow \varphi$ of T fails (an easy task left as an exercise), for then by Soundness $\mathbf{KD} \not\vdash \Box\varphi \rightarrow \varphi$. \square

Proposition prf.2. $\mathbf{KB} \neq \mathbf{K4}$.

Proof. We construct a symmetric model where some instance of 4 fails; since obviously the instance is derivable for $\mathbf{K4}$ but not in \mathbf{KB} , it will follow $\mathbf{K4} \not\subseteq \mathbf{KB}$. Consider the symmetric model \mathfrak{M} of Figure 1 Since the model is symmetric, K and B are true in \mathfrak{M} (by ?? and ??, respectively). However, $\mathfrak{M}, w \not\models \Box p \rightarrow \Box\Box p$. \square

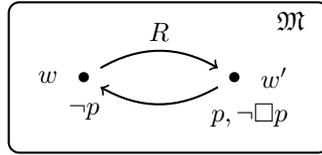


Figure 1: A symmetric model falsifying an instance of 4.

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thm:KTBnot45

Theorem prf.3. $\mathbf{KTB} \not\vdash 4$ and $\mathbf{KTB} \not\vdash 5$.

Proof. By ?? we know that all instances of T and B are true in each reflexive symmetric model (respectively). So by Soundness it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figure Figure 2. Then $\mathfrak{M}, w_1 \not\models \Box p \rightarrow \Box\Box p$, so the instance of 4 with $\varphi = p$ fails at w_1 . Similarly, $\mathfrak{M}, w_2 \not\models \Diamond\neg p \rightarrow \Box\Diamond\neg p$, so the instance of 5 with $\varphi = \neg p$ fails at w_2 . \square

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Theorem prf.4. $\mathbf{KD5} \neq \mathbf{KT4} = \mathbf{S4}$.

Proof. By ?? we know that all instances of D and 5 to be true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figure 3, and notice that $\mathfrak{M}, w_1 \not\models \Box p \rightarrow \Box\Box p$. \square

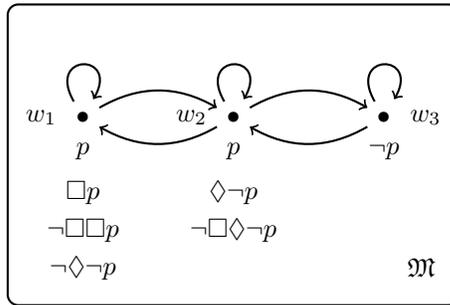


Figure 2: The model for Theorem prf.3.

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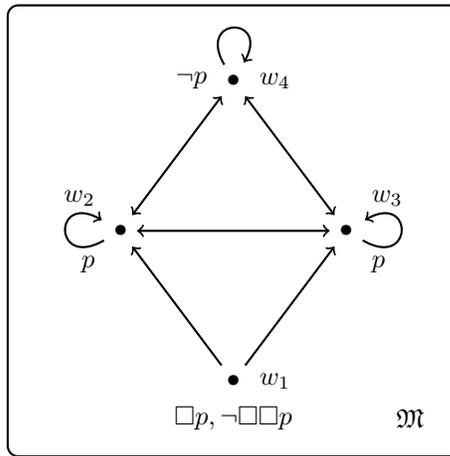


Figure 3: The model for Theorem prf.4.

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Problem prf.1. Give an alternative proof of Theorem prf.4 using a model with 3 worlds.

Problem prf.2. Provide a single reflexive transitive model showing that both $\mathbf{KT4} \not\vdash \mathbf{B}$ and $\mathbf{KT4} \not\vdash 5$.

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Bibliography