

## prf.1 Showing Systems are Distinct

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sec

In ?? we saw how to prove that two systems of modal logic are in fact the same system. ?? allows us to show that two modal systems  $\Sigma$  and  $\Sigma'$  are distinct, by finding a formula  $\varphi$  such that  $\Sigma' \vdash \varphi$  that fails in a model of  $\Sigma$ .

### Proposition prf.1. $\mathbf{KD} \subsetneq \mathbf{KT}$

*Proof.* This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show  $\mathbf{KD} \subseteq \mathbf{KT}$  we need to see that  $\mathbf{KD} \vdash \psi$  implies  $\mathbf{KT} \vdash \psi$ , which follows from  $\mathbf{KT} \vdash \mathbf{D}$ , as shown in ????. To show that the inclusion is proper, by Soundness (??), it suffices to exhibit a model of  $\mathbf{KD}$  where T, i.e.,  $\Box p \rightarrow p$ , fails (an easy task left as an exercise), for then by Soundness  $\mathbf{KD} \not\vdash \Box p \rightarrow p$ .  $\square$

### Proposition prf.2. $\mathbf{KB} \neq \mathbf{K4}$ .

*Proof.* We construct a symmetric model where some instance of 4 fails; since obviously the instance is derivable for  $\mathbf{K4}$  but not in  $\mathbf{KB}$ , it will follow  $\mathbf{K4} \not\subseteq \mathbf{KB}$ . Consider the symmetric model  $\mathfrak{M}$  of Figure 1. Since the model is symmetric, K and B are true in  $\mathfrak{M}$  (by ?? and ??, respectively). However,  $\mathfrak{M}, w_1 \not\vdash \Box p \rightarrow \Box \Box p$ .  $\square$

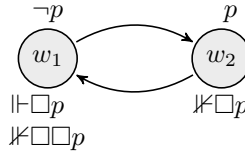


Figure 1: A symmetric model falsifying an instance of 4.

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thm:KTBnot45

### Theorem prf.3. $\mathbf{KTB} \not\vdash 4$ and $\mathbf{KTB} \not\vdash 5$ .

*Proof.* By ?? we know that all instances of T and B are true in every reflexive symmetric model (respectively). So by soundness, it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figure 2. Then  $\mathfrak{M}, w_1 \not\vdash \Box p \rightarrow \Box \Box p$ , so 4 fails at  $w_1$ . Similarly,  $\mathfrak{M}, w_2 \not\vdash \Diamond \neg p \rightarrow \Box \Diamond \neg p$ , so the instance of 5 with  $\varphi = \neg p$  fails at  $w_2$ .  $\square$

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thm:KD5not4

### Theorem prf.4. $\mathbf{KD5} \neq \mathbf{KT4} = \mathbf{S4}$ .

*Proof.* By ?? we know that all instances of D and 5 are true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figure 3, and notice that  $\mathfrak{M}, w_1 \not\vdash \Box p \rightarrow \Box \Box p$ .  $\square$

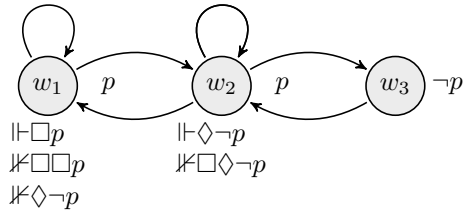


Figure 2: The model for [Theorem prf.3](#).

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fig:KTBnot45

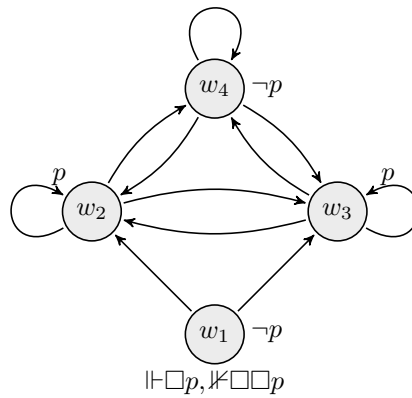


Figure 3: The model for [Theorem prf.4](#).

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**Problem prf.1.** Give an alternative proof of [Theorem prf.4](#) using a model with 3 worlds.

**Problem prf.2.** Provide a single reflexive transitive model showing that both  $\text{KT4} \not\vdash \text{B}$  and  $\text{KT4} \not\vdash 5$ .

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## Bibliography