prf.1 Soundness

mod:prf:snd:sec

mod:prf:snd:thm:soundness

Theorem prf.1 (Soundness Theorem). *If schemas $\varphi_1, \ldots, \varphi_n$ are valid in the classes of models $\mathcal{C}_n, \ldots, \mathcal{C}_n$, respectively, then $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$ implies that ψ is valid in the class of models $\mathcal{C}_n \cap \cdots \cap \mathcal{C}_n$.*

Proof. By induction on length of proofs. For brevity, put $\mathcal{C} = \mathcal{C}_n \cap \cdots \cap \mathcal{C}_n$.

1. Induction Basis: If ψ has a proof of length 1, then it is either a tautological instance or an instance of K, or an instance of one of the schemas. In the first case, ψ is valid in $\mathcal{C}$, since tautological instance are valid in *any* class of models, by **??**. Similarly in the second case, by **??**. Finally in the third case, since ψ is valid in $\mathcal{C}_i$ and $\mathcal{C} \subseteq \mathcal{C}_i$, we have that ψ is valid in $\mathcal{C}$ as well.

2. Inductive step: Suppose ψ has a proof of length $k > 1$. If ψ is a tautological instance or an instance of one of the schemas, we proceed as in the previous step. So suppose ψ is obtained by MP from previous formulas $\chi \to \psi$ and χ. Then $\chi \to \psi$ and χ have proofs of length $< k$, and by inductive hypothesis they are valid in $\mathcal{C}$. By **??**, ψ is valid in $\mathcal{C}$ as well. Finally suppose ψ is obtained by NEC from χ (so that $\psi = \Box\chi$). By inductive hypothesis, χ is valid in $\mathcal{C}$, and by **??** so is φ. □

Photo Credits

Bibliography