

## prf.1 Soundness

mod:prf:snd:  
sec A derivation system is called sound if everything that can be derived is valid. When considering modal systems, i.e., derivations where in addition to K we can use instances of some formulas  $\varphi_1, \dots, \varphi_n$ , we want every derivable formula to be true in any model in which  $\varphi_1, \dots, \varphi_n$  are true.

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thm:soundness **Theorem prf.1** (Soundness Theorem). *If every instance of  $\varphi_1, \dots, \varphi_n$  is valid in the classes of models  $\mathcal{C}_1, \dots, \mathcal{C}_n$ , respectively, then  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$  implies that  $\psi$  is valid in the class of models  $\mathcal{C}_1 \cap \dots \cap \mathcal{C}_n$ .*

*Proof.* By induction on length of proofs. For brevity, put  $\mathcal{C} = \mathcal{C}_1 \cap \dots \cap \mathcal{C}_n$ .

1. Induction Basis: If  $\psi$  has a proof of length 1, then it is either a tautological instance, an instance of K, or of DUAL, or an instance of one of  $\varphi_1, \dots, \varphi_n$ . In the first case,  $\psi$  is valid in  $\mathcal{C}$ , since tautological instances are valid in any class of models, by ???. Similarly in the second case, by ?? and ??. Finally in the third case, since  $\psi$  is valid in  $\mathcal{C}_i$  and  $\mathcal{C} \subseteq \mathcal{C}_i$ , we have that  $\psi$  is valid in  $\mathcal{C}$  as well.
2. Inductive step: Suppose  $\psi$  has a proof of length  $k > 1$ . If  $\psi$  is a tautological instance or an instance of one of  $\varphi_1, \dots, \varphi_n$ , we proceed as in the previous step. So suppose  $\psi$  is obtained by MP from previous formulas  $\chi \rightarrow \psi$  and  $\chi$ . Then  $\chi \rightarrow \psi$  and  $\chi$  have proofs of length  $< k$ , and by inductive hypothesis they are valid in  $\mathcal{C}$ . By ??,  $\psi$  is valid in  $\mathcal{C}$  as well. Finally suppose  $\psi$  is obtained by NEC from  $\chi$  (so that  $\psi = \Box\chi$ ). By inductive hypothesis,  $\chi$  is valid in  $\mathcal{C}$ , and by ?? so is  $\psi$ .  $\square$

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## Bibliography