prf.1 Soundness

mod:prf:snd:sec

A derivation system is called sound if everything that can be derived is valid. When considering modal systems, i.e., derivations where in addition to K we can use instances of some formulas $\varphi_1, \ldots, \varphi_n$, we want every derivable formula to be true in any model in which $\varphi_1, \ldots, \varphi_n$ are true.

mod:prf:snd:thm:soundness

Theorem prf.1 (Soundness Theorem)**.** *If every instance of $\varphi_1, \ldots, \varphi_n$ is valid in the classes of models $\mathcal{C}_1, \ldots, \mathcal{C}_n$, respectively, then $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$ implies that ψ is valid in the class of models $\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_n$.*

Proof. By induction on length of proofs. For brevity, put $\mathcal{C} = \mathcal{C}_n \cap \cdots \cap \mathcal{C}_n$.

1. Induction Basis: If ψ has a proof of length 1, then it is either a tautological instance, an instance of K, or of DUAL, or an instance of one of $\varphi_1, \ldots, \varphi_n$. In the first case, ψ is valid in $\mathcal{C}$, since tautological instance are valid in *any* class of models, by **??**. Similarly in the second case, by **??** and **??**. Finally in the third case, since ψ is valid in $\mathcal{C}_i$ and $\mathcal{C} \subseteq \mathcal{C}_i$, we have that ψ is valid in $\mathcal{C}$ as well.

2. Inductive step: Suppose ψ has a proof of length $k > 1$. If ψ is a tautological instance or an instance of one of $\varphi_1, \ldots, \varphi_n$, we proceed as in the previous step. So suppose ψ is obtained by MP from previous formulas $\chi \to \psi$ and χ. Then $\chi \to \psi$ and χ have proofs of length $< k$, and by inductive hypothesis they are valid in $\mathcal{C}$. By **??**, ψ is valid in $\mathcal{C}$ as well. Finally suppose ψ is obtained by NEC from χ (so that $\psi = \Box\chi$). By inductive hypothesis, χ is valid in $\mathcal{C}$, and by **??** so is ψ. □

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Bibliography