prf.1  Soundness

A derivation system is called sound if everything that can be derived is valid. When considering modal systems, i.e., derivations where in addition to K we can use instances of some formulas $\varphi_1, \ldots, \varphi_n$, we want every derivable formula to be true in any model in which $\varphi_1, \ldots, \varphi_n$ are true.

Theorem prf.1 (Soundness Theorem). If every instance of $\varphi_1, \ldots, \varphi_n$ is valid in the classes of models $C_1, \ldots, C_n$, respectively, then $K\varphi_1 \ldots \varphi_n \vdash \psi$ implies that $\psi$ is valid in the class of models $C_1 \cap \cdots \cap C_n$.

Proof. By induction on length of proofs. For brevity, put $C = C_1 \cap \cdots \cap C_n$.

1. Induction Basis: If $\psi$ has a proof of length 1, then it is either a tautological instance, an instance of K, or of DUAL, or an instance of one of $\varphi_1, \ldots, \varphi_n$. In the first case, $\psi$ is valid in $C$, since tautological instance are valid in any class of models, by ???. Similarly in the second case, by ?? and ???. Finally in the third case, since $\psi$ is valid in $C_i$ and $C \subseteq C_i$, we have that $\psi$ is valid in $C$ as well by ??.

2. Inductive step: Suppose $\psi$ has a proof of length $k > 1$. If $\psi$ is a tautological instance or an instance of one of $\varphi_1, \ldots, \varphi_n$, we proceed as in the previous step. So suppose $\psi$ is obtained by MP from previous formulas $\chi \rightarrow \psi$ and $\chi$. Then $\chi \rightarrow \psi$ and $\chi$ have proofs of length < $k$, and by inductive hypothesis they are valid in $C$. By ??, $\psi$ is valid in $C$ as well. Finally suppose $\psi$ is obtained by NEC from $\chi$ (so that $\psi = \Box \chi$). By inductive hypothesis, $\chi$ is valid in $C$, and by ?? so is $\psi$. 

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Bibliography