prf.1  Soundness

A derivation system is called sound if everything that can be derived is valid. When considering modal systems, i.e., derivations where in addition to \(K\) we can use instances of some formulas \(\varphi_1, \ldots, \varphi_n\), we want every derivable formula to be true in any model in which \(\varphi_1, \ldots, \varphi_n\) are true.

Theorem prf.1 (Soundness Theorem). If every instance of \(\varphi_1, \ldots, \varphi_n\) is valid in the classes of models \(C_1, \ldots, C_n\), respectively, then \(K\varphi_1 \ldots \varphi_n \vdash \psi\) implies that \(\psi\) is valid in the class of models \(C_1 \cap \cdots \cap C_n\).

Proof. By induction on length of proofs. For brevity, put \(C = C_n \cap \cdots \cap C_1\).

1. Induction Basis: If \(\psi\) has a proof of length 1, then it is either a tautological instance, an instance of \(K\), or of \(\text{dual}\), or an instance of one of \(\varphi_1, \ldots, \varphi_n\). In the first case, \(\psi\) is valid in \(C\), since tautological instance are valid in any class of models, by ?? . Similarly in the second case, by ?? and ?? . Finally in the third case, since \(\psi\) is valid in \(C_i\) and \(C \subseteq C_i\), we have that \(\psi\) is valid in \(C\) as well.

2. Inductive step: Suppose \(\psi\) has a proof of length \(k > 1\). If \(\psi\) is a tautological instance or an instance of one of \(\varphi_1, \ldots, \varphi_n\), we proceed as in the previous step. So suppose \(\psi\) is obtained by \(\text{mp}\) from previous formulas \(\chi \rightarrow \psi\) and \(\chi\). Then \(\chi \rightarrow \psi\) and \(\chi\) have proofs of length < \(k\), and by inductive hypothesis they are valid in \(C\). By ?? , \(\psi\) is valid in \(C\) as well. Finally suppose \(\psi\) is obtained by \(\text{ne}\) from \(\chi\) (so that \(\psi = \Box \chi\)). By inductive hypothesis, \(\chi\) is valid in \(C\), and by ?? so is \(\psi\).

\[\square\]

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Bibliography