Properties of Derivability

**Proposition prf.1.** Let $\Sigma$ be a modal system and $\Gamma$ a set of modal formulas. The following properties hold:

1. Monotonicity: If $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_\Sigma \varphi$;
2. Reflexivity: If $\varphi \in \Gamma$ then $\Gamma \vdash_\Sigma \varphi$;
3. Cut: If $\Gamma \vdash_\Sigma \varphi$ and $\Delta \cup \{\varphi\} \vdash_\Sigma \psi$ then $\Gamma \cup \Delta \vdash_\Sigma \psi$;
4. Deduction theorem: $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi$ if and only if $\Gamma \vdash_\Sigma \psi \to \varphi$;
5. $\Gamma \vdash_\Sigma \varphi_1$ and ... and $\Gamma \vdash_\Sigma \varphi_n$ and $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \psi)\cdots)$ is a tautological instance, then $\Gamma \vdash_\Sigma \psi$.

The proof is an easy exercise. Part (5) of Proposition prf.1 gives us that, for instance, if $\Gamma \vdash_\Sigma \varphi \lor \psi$ and $\Gamma \vdash_\Sigma \neg \varphi$, then $\Gamma \vdash_\Sigma \psi$. Also, in what follows, we write $\Gamma, \varphi \vdash_\Sigma \psi$ instead of $\Gamma \cup \{\varphi\} \vdash_\Sigma \psi$.

**Definition prf.2.** A set $\Gamma$ is *deductively closed* relatively to a system $\Sigma$ if and only if $\Gamma \vdash_\Sigma \varphi$ implies $\varphi \in \Gamma$.

Photo Credits

Bibliography