## prf.1 Proofs in Modal Systems

nml:prf:prs: sec

We now come to proofs in systems of modal logic other than **K**.

nml:prf:prs: prop:S5facts

**Proposition prf.1.** *The following provability results obtain:*

1. $\mathbf{KT5} \vdash \mathrm{B}$;
2. $\mathbf{KT5} \vdash 4$;
3. $\mathbf{KDB4} \vdash \mathrm{T}$;
4. $\mathbf{KB4} \vdash 5$;
5. $\mathbf{KB5} \vdash 4$;
6. $\mathbf{KT} \vdash \mathrm{D}$. nml:prf:prs: prop:S5facts-KT-D

*Proof.* We exhibit proofs for each.

1. $\mathbf{KT5} \vdash \mathrm{B}$:

| | | |
|---|---|---|
| 1. | $\mathbf{KT5} \vdash \Diamond\varphi \to \Box\Diamond\varphi$ | 5 |
| 2. | $\mathbf{KT5} \vdash \varphi \to \Diamond\varphi$ | $\mathrm{T}_\Diamond$ |
| 3. | $\mathbf{KT5} \vdash \varphi \to \Box\Diamond\varphi$ | PL. |

2. $\mathbf{KT5} \vdash 4$:

| | | |
|---|---|---|
| 1. | $\mathbf{KT5} \vdash \Diamond\Box\varphi \to \Box\Diamond\Box\varphi$ | 5 with $\Box\varphi$ for $p$ |
| 2. | $\mathbf{KT5} \vdash \Box\varphi \to \Diamond\Box\varphi$ | $\mathrm{T}_\Diamond$ with $\Box\varphi$ for $p$ |
| 3. | $\mathbf{KT5} \vdash \Box\varphi \to \Box\Diamond\Box\varphi$ | PL, 1, 2 |
| 4. | $\mathbf{KT5} \vdash \Diamond\Box\varphi \to \Box\varphi$ | $5_\Diamond$ |
| 5. | $\mathbf{KT5} \vdash \Box\Diamond\Box\varphi \to \Box\Box\varphi$ | RK, 4 |
| 6. | $\mathbf{KT5} \vdash \Box\varphi \to \Box\Box\varphi$ | PL, 3, 5. |

3. $\mathbf{KDB4} \vdash \mathrm{T}$:

| | | |
|---|---|---|
| 1. | $\mathbf{KDB4} \vdash \Diamond\Box\varphi \to \varphi$ | $\mathrm{B}_\Diamond$ |
| 2. | $\mathbf{KDB4} \vdash \Box\Box\varphi \to \Diamond\Box\varphi$ | D with $\Box\varphi$ for $p$ |
| 3. | $\mathbf{KDB4} \vdash \Box\Box\varphi \to \varphi$ | PL1, 2 |
| 4. | $\mathbf{KDB4} \vdash \Box\varphi \to \Box\Box\varphi$ | 4 |
| 5. | $\mathbf{KDB4} \vdash \Box\varphi \to \varphi$ | PL, 1, 4. |

4. $\mathbf{KB4} \vdash 5$:

| | | |
|---|---|---|
| 1. | $\mathbf{KB4} \vdash \Diamond\varphi \to \Box\Diamond\Diamond\varphi$ | B with $\Diamond\varphi$ for $p$ |
| 2. | $\mathbf{KB4} \vdash \Diamond\Diamond\varphi \to \Diamond\varphi$ | $4_\Diamond$ |
| 3. | $\mathbf{KB4} \vdash \Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi$ | RK, 2 |
| 4. | $\mathbf{KB4} \vdash \Diamond\varphi \to \Box\Diamond\varphi$ | PL, 1, 3. |

5. $\mathbf{KB5} \vdash 4$:

   1. $\mathbf{KB5} \vdash \Box\varphi \to \Box\Diamond\Box\varphi$ — B with $\Box\varphi$ for $p$
   2. $\mathbf{KB5} \vdash \Diamond\Box\varphi \to \Box\varphi$ — $5_\Diamond$
   3. $\mathbf{KB5} \vdash \Box\Diamond\Box\varphi \to \Box\Box\varphi$ — RK, 2
   4. $\mathbf{KB5} \vdash \Box\varphi \to \Box\Box\varphi$ — PL, 1, 3.

6. $\mathbf{KT} \vdash \mathrm{D}$:

   1. $\mathbf{KT} \vdash \Box\varphi \to \varphi$ — T
   2. $\mathbf{KT} \vdash \varphi \to \Diamond\varphi$ — $\mathrm{T}_\Diamond$
   3. $\mathbf{KT} \vdash \Box\varphi \to \Diamond\varphi$ — PL, 1, 2 □

**Definition prf.2.** Following tradition, we define **S4** to be the system **KT4**, and **S5** the system **KTB4**.

The following proposition shows that the classical system **S5** has several equivalent axiomatizations. This should not surprise, as the various combinations of axioms all characterize equivalence relations (see **??**).

**Proposition prf.3.** $\mathbf{KTB4} = \mathbf{KT5} = \mathbf{KDB4} = \mathbf{KDB5}$. *nml:prf:prs:prop:S5*

*Proof.* Exercise. □

**Problem prf.1.** Prove Proposition prf.3.

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## Bibliography