prf.1 Proofs in Modal Systems

mod:prf:prs: sec

We now come to proofs in systems of modal logic other than **K**.

mod:prf:prs: prop:S5facts

Proposition prf.1. *The following provability results obtain:*

1. $\mathbf{KT5} \vdash$ B;
2. $\mathbf{KT5} \vdash$ 4;
3. $\mathbf{KDB4} \vdash$ T;
4. $\mathbf{KB4} \vdash$ 5;
5. $\mathbf{KB5} \vdash$ 4;
6. $\mathbf{KT} \vdash$ D. (mod:prf:prs: prop:S5facts-KT-D)

Proof. We exhibit proofs for each.

1. $\mathbf{KT5} \vdash$ B:

1.	$\mathbf{KT5} \vdash \Diamond\varphi \to \Box\Diamond\varphi$	5
2.	$\mathbf{KT5} \vdash \varphi \to \Diamond\varphi$	$\mathrm{T}_\Diamond$
3.	$\mathbf{KT5} \vdash \varphi \to \Box\Diamond\varphi$	PL.

2. $\mathbf{KT5} \vdash$ 4:

1.	$\mathbf{KT5} \vdash \Diamond\Box\varphi \to \Box\Diamond\Box\varphi$	5 with $\Box\varphi$ for p
2.	$\mathbf{KT5} \vdash \Box\varphi \to \Diamond\Box\varphi$	$\mathrm{T}_\Diamond$ with $\Box\varphi$ for p
3.	$\mathbf{KT5} \vdash \Box\varphi \to \Box\Diamond\Box\varphi$	PL, 1, 2
4.	$\mathbf{KT5} \vdash \Diamond\Box\varphi \to \Box\varphi$	$5_\Diamond$
5.	$\mathbf{KT5} \vdash \Box\Diamond\Box\varphi \to \Box\Box\varphi$	RK, 4
6.	$\mathbf{KT5} \vdash \Box\varphi \to \Box\Box\varphi$	PL, 3, 5.

3. $\mathbf{KDB4} \vdash$ T:

1.	$\mathbf{KDB4} \vdash \Diamond\Box\varphi \to \varphi$	$\mathrm{B}_\Diamond$
2.	$\mathbf{KDB4} \vdash \Box\Box\varphi \to \Diamond\Box\varphi$	D with $\Box\varphi$ for p
3.	$\mathbf{KDB4} \vdash \Box\Box\varphi \to \varphi$	PL1, 2
4.	$\mathbf{KDB4} \vdash \Box\varphi \to \Box\Box\varphi$	4
5.	$\mathbf{KDB4} \vdash \Box\varphi \to \varphi$	PL, 1, 4.

4. $\mathbf{KB4} \vdash$ 5:

1.	$\mathbf{KB4} \vdash \Diamond\varphi \to \Box\Diamond\Diamond\varphi$	B with $\Diamond\varphi$ for p
2.	$\mathbf{KB4} \vdash \Diamond\Diamond\varphi \to \Diamond\varphi$	$4_\Diamond$
3.	$\mathbf{KB4} \vdash \Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi$	RK, 2
4.	$\mathbf{KB4} \vdash \Diamond\varphi \to \Box\Diamond\varphi$	PL, 1, 3.

5. **KB5** $\vdash$ 4:

1.	**KB5** $\vdash \Box\varphi \to \Box\Diamond\Box\varphi$	B with $\Box\varphi$ for p
2.	**KB5** $\vdash \Diamond\Box\varphi \to \Box\varphi$	$5_\Diamond$
3.	**KB5** $\vdash \Box\Diamond\Box\varphi \to \Box\Box\varphi$	RK, 2
4.	**KB5** $\vdash \Box\varphi \to \Box\Box\varphi$	PL, 1, 3.

6. **KT** $\vdash$ D:

1.	**KT** $\vdash \Box\varphi \to \varphi$	T
2.	**KT** $\vdash \varphi \to \Diamond\varphi$	$\mathrm{T}_\Diamond$
3.	**KT** $\vdash \Box\varphi \to \Diamond\varphi$	PL, 1, 2

□

Definition prf.2. Following tradition, we define **S4** to be the system **KT4**, and **S5** the system **KTB4**.

The following proposition shows that the classical system **S5** has several equivalent axiomatizations. This should not surprise, as the various combinations of axioms all characterize equivalence relations (see **??**).

Proposition prf.3. **KTB4** = **KT5** = **KDB4** = **KDB5**. *mod:prf:prs:prop:S5*

Proof. Exercise. □

Problem prf.1. Prove Proposition prf.3.

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Bibliography