

prf.1 Proofs in Modal Systems

mod:prf:prs:
sec

We now come to proofs in systems of modal logic other than **K**.

mod:prf:prs:
prop:S5facts

Proposition prf.1. *The following provability results obtain:*

1. **KT5** \vdash B;
2. **KT5** \vdash 4;
3. **KDB4** \vdash T;
4. **KB4** \vdash 5;
5. **KB5** \vdash 4;
6. **KT** \vdash D.

mod:prf:prs:
prop:S5facts-KT-D

Proof. We exhibit proofs for each.

1. **KT5** \vdash B:

1. **KT5** $\vdash \Diamond\varphi \rightarrow \Box\Diamond\varphi$ 5
2. **KT5** $\vdash \varphi \rightarrow \Diamond\varphi$ T_\Diamond
3. **KT5** $\vdash \varphi \rightarrow \Box\Diamond\varphi$ PL.

2. **KT5** \vdash 4:

1. **KT5** $\vdash \Diamond\Box\varphi \rightarrow \Box\Diamond\Box\varphi$ 5 with $\Box\varphi$ for p
2. **KT5** $\vdash \Box\varphi \rightarrow \Diamond\Box\varphi$ T_\Diamond with $\Box\varphi$ for p
3. **KT5** $\vdash \Box\varphi \rightarrow \Box\Diamond\Box\varphi$ PL, 1, 2
4. **KT5** $\vdash \Diamond\Box\varphi \rightarrow \Box\varphi$ 5_\Diamond
5. **KT5** $\vdash \Box\Diamond\Box\varphi \rightarrow \Box\Box\varphi$ RK, 4
6. **KT5** $\vdash \Box\varphi \rightarrow \Box\Box\varphi$ PL, 3, 5.

3. **KDB4** \vdash T:

1. **KDB4** $\vdash \Diamond\Box\varphi \rightarrow \varphi$ B_\Diamond
2. **KDB4** $\vdash \Box\Box\varphi \rightarrow \Diamond\Box\varphi$ D with $\Box\varphi$ for p
3. **KDB4** $\vdash \Box\Box\varphi \rightarrow \varphi$ PL1, 2
4. **KDB4** $\vdash \Box\varphi \rightarrow \Box\Box\varphi$ 4
5. **KDB4** $\vdash \Box\varphi \rightarrow \varphi$ PL, 1, 4.

4. **KB4** \vdash 5:

1. **KB4** $\vdash \Diamond\varphi \rightarrow \Box\Diamond\Diamond\varphi$ B with $\Diamond\varphi$ for p
2. **KB4** $\vdash \Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ 4_\Diamond
3. **KB4** $\vdash \Box\Diamond\Diamond\varphi \rightarrow \Box\Diamond\varphi$ RK, 2
4. **KB4** $\vdash \Diamond\varphi \rightarrow \Box\Diamond\varphi$ PL, 1, 3.

5. **KB5** \vdash 4:

1. **KB5** $\vdash \Box\varphi \rightarrow \Box\Diamond\Box\varphi$ B with $\Box\varphi$ for p
2. **KB5** $\vdash \Diamond\Box\varphi \rightarrow \Box\varphi$ 5_\Diamond
3. **KB5** $\vdash \Box\Diamond\Box\varphi \rightarrow \Box\Box\varphi$ RK, 2
4. **KB5** $\vdash \Box\varphi \rightarrow \Box\Box\varphi$ PL, 1, 3.

6. **KT** \vdash D:

1. **KT** $\vdash \Box\varphi \rightarrow \varphi$ T
2. **KT** $\vdash \varphi \rightarrow \Diamond\varphi$ T_\Diamond
3. **KT** $\vdash \Box\varphi \rightarrow \Diamond\varphi$ PL, 1, 2

□

Definition prf.2. Following tradition, we define **S4** to be the system **KT4**, and **S5** the system **KTB4**.

The following proposition shows that the classical system **S5** has several equivalent axiomatizations. This should not surprise, as the various combinations of axioms all characterize equivalence relations (see ??).

Proposition prf.3. **KTB4 = KT5 = KDB4 = KDB5.**

*mod:prf:prs:
prop:S5*

Proof. Exercise.

□

Problem prf.1. Prove [Proposition prf.3](#).

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Bibliography