prf.1 Proofs in K

mod:prf:prk:sec

In order to practice proofs in the smallest modal system, we show the valid formulas on the left-hand side of ?? can all be given **K**-proofs.

Proposition prf.1. $\mathbf{K} \vdash \Box\varphi \to \Box(\psi \to \varphi)$

Proof.

1.	$\varphi \to (\psi \to \varphi)$	TAUT
2.	$\Box(\varphi \to (\psi \to \varphi))$	NEC, 1
3.	$\Box(\varphi \to (\psi \to \varphi)) \to (\Box\varphi \to \Box(\psi \to \varphi))$	K
4.	$\Box\varphi \to \Box(\psi \to \varphi)$	MP, 2, 3

□

Proposition prf.2. $\mathbf{K} \vdash \Box(\varphi \land \psi) \to (\Box\varphi \land \Box\psi)$

Proof.

1.	$(\varphi \land \psi) \to \varphi$	TAUT
2.	$\Box((\varphi \land \psi) \to \varphi)$	NEC
3.	$\Box((\varphi \land \psi) \to \varphi) \to (\Box(\varphi \land \psi) \to \Box\varphi)$	K
4.	$\Box(\varphi \land \psi) \to \Box\varphi$	MP, 2, 3
5.	$(\varphi \land \psi) \to \psi$	TAUT
6.	$\Box((\varphi \land \psi) \to \psi)$	NEC
7.	$\Box((\varphi \land \psi) \to \psi) \to (\Box(\varphi \land \psi) \to \Box\psi)$	K
8.	$\Box(\varphi \land \psi) \to \Box\psi$	MP, 6, 7
9.	$(\Box(\varphi \land \psi) \to \Box\varphi) \to$ $((\Box(\varphi \land \psi) \to \Box\varphi) \to$ $(\Box(\varphi \land \psi) \to (\Box\varphi \land \Box\psi)))$	TAUT
10.	$(\Box(\varphi \land \psi) \to \Box\varphi) \to$ $(\Box(\varphi \land \psi) \to (\Box\varphi \land \Box\psi))$	MP, 4, 9
11.	$\Box(\varphi \land \psi) \to (\Box\varphi \land \Box\psi)$	MP, 4, 10.

Note that the formula on line 9 is an instance of the tautology

$$(p \to q) \to ((p \to r) \to (p \to (q \land r))).$$

□

Proposition prf.3. $\mathbf{K} \vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$

Proof.

1. $\varphi \to (\psi \to (\varphi \land \psi))$ — TAUT
2. $\Box(\varphi \to (\psi \to (\varphi \land \psi)))$ — NEC, 1
3. $\Box(\varphi \to (\psi \to (\varphi \land \psi))) \to (\Box\varphi \to \Box(\psi \to (\varphi \land \psi)))$ — K
4. $\Box\varphi \to \Box(\psi \to (\varphi \land \psi))$ — MP, 2, 3
5. $\Box(\psi \to (\varphi \land \psi)) \to (\Box\psi \to \Box(\varphi \land \psi))$ — K
6. $(\Box\varphi \to \Box(\psi \to (\varphi \land \psi))) \to$ $(\Box(\psi \to (\varphi \land \psi)) \to (\Box\psi \to \Box(\varphi \land \psi))) \to$ $(\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi))))$ — TAUT
7. $(\Box(\psi \to (\varphi \land \psi)) \to (\Box\psi \to \Box(\varphi \land \psi))) \to$ $(\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi)))$ — MP, 4, 6
8. $\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi)))$ — MP, 5, 7
9. $(\Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi)))) \to$ $((\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi))$ — TAUT
10. $(\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$ — MP, 8, 9

The formulas on lines 6 and 9 are instances of the tautologies

$$(p \to q) \to ((q \to r) \to (p \to r))$$
$$(p \to (q \to r)) \to ((p \land q) \to r)$$

□

Proposition prf.4. $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$

Proof.

1. $\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$ — DUAL
2. $(\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p) \to$ $(\neg\Box\neg\neg p \to \Diamond\neg p)$ — TAUT
3. $\neg\Box\neg\neg p \to \Diamond\neg p$ — MP, 1, 2
4. $\neg\neg p \to p$ — TAUT
5. $\Box(\neg\neg p \to p)$ — NEC, 4
6. $\Box(\neg\neg p \to p) \to (\Box\neg\neg p \to \Box p)$ — K
7. $(\Box\neg\neg p \to \Box p)$ — MP, 5, 6
8. $(\Box\neg\neg p \to \Box p) \to (\neg\Box p \to \neg\Box\neg\neg p)$ — TAUT
9. $\neg\Box p \to \neg\Box\neg\neg p$ — MP, 7, 8
10. $(\neg\Box p \to \neg\Box\neg\neg p) \to$ $((\neg\Box\neg\neg p \to \Diamond\neg p) \to (\neg\Box p \to \Diamond\neg p))$ — TAUT
11. $(\neg\Box\neg\neg p \to \Diamond\neg p) \to (\neg\Box p \to \Diamond\neg p)$ — MP, 9, 10
12. $\neg\Box p \to \Diamond\neg p$ — MP, 3, 11

The formulas on lines 8 and 10 are instances of the tautologies

$$(p \to q) \to (\neg q \to \neg p)$$
$$(p \to q) \to ((q \to r) \to (p \to r)).$$

□

Problem prf.1. Find derivations in $\mathbf{K}$ for the following formulas:

1. $\Box\neg p \to \Box(p \to q)$
2. $(\Box p \vee \Box q) \to \Box(p \vee q)$
3. $\Diamond p \to \Diamond(p \vee q)$

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Bibliography