

## prf.1 Proofs in K

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sec

In order to practice proofs in the smallest modal system, we show the valid formulas on the left-hand side of the table of ?? can all be given **K**-proofs. Justifications for steps that are either tautological instances or follow by tautological implication from previous one are just marked “PL” (for “Propositional Logic”).

**Proposition prf.1.**  $\mathbf{K} \vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$

*Proof.*

- |    |   |             |
|----|---|-------------|
| 1. | $(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$                                 | PL          |
| 2. | $\Box[(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)]$                           | NEC         |
| 3. | $\Box(\varphi \rightarrow \psi) \rightarrow \Box(\neg\psi \rightarrow \neg\varphi)$                         | K, MP       |
| 4. | $\Box(\neg\psi \rightarrow \neg\varphi) \rightarrow (\Box\neg\psi \rightarrow \Box\neg\varphi)$             | K           |
| 5. | $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\neg\psi \rightarrow \Box\neg\varphi)$                     | PL, 3,4     |
| 6. | $(\Box\neg\psi \rightarrow \Box\neg\varphi) \rightarrow (\neg\Box\neg\varphi \rightarrow \neg\Box\neg\psi)$ | PL          |
| 7. | $\Box(\varphi \rightarrow \psi) \rightarrow (\neg\Box\neg\varphi \rightarrow \neg\Box\neg\psi)$             | PL, 5, 6    |
| 8. | $\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$                     | re-writing. |

□

**Proposition prf.2.**  $\mathbf{K} \vdash \Box\varphi \rightarrow (\Diamond(\varphi \rightarrow \psi) \rightarrow \Diamond\psi)$

*Proof.*

- |    |   |             |
|----|---|-------------|
| 1. | $\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \rightarrow \psi))$                     | PL          |
| 2. | $\Box[\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \rightarrow \psi))]$               | NEC         |
| 3. | $\Box\varphi \rightarrow \Box(\neg\psi \rightarrow \neg(\varphi \rightarrow \psi))$             | K           |
| 4. | $\Box\varphi \rightarrow (\Box\neg\psi \rightarrow \Box\neg(\varphi \rightarrow \psi))$         | K           |
| 5. | $\Box\varphi \rightarrow (\neg\Box\neg(\varphi \rightarrow \psi) \rightarrow \neg\Box\neg\psi)$ | PL          |
| 6. | $\Box\varphi \rightarrow (\Diamond(\varphi \rightarrow \psi) \rightarrow \Diamond\psi)$         | re-writing. |

□

**Proposition prf.3.**  $\mathbf{K} \vdash \Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

*Proof.*

- |    |   |            |
|----|---|------------|
| 1. | $\neg(\varphi \rightarrow \neg\psi) \rightarrow \varphi$              | PL         |
| 2. | $\Box\neg(\varphi \rightarrow \neg\psi) \rightarrow \Box\varphi$      | NEC, K     |
| 3. | $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi$                   | re-writing |
| 4. | $\Box(\varphi \wedge \psi) \rightarrow \Box\psi$                      | similarly  |
| 5. | $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$ | PL, 3,4.   |

□

**Proposition prf.4.**  $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

*Proof.*

1.  $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$  PL
2.  $\Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))$  NEC, K
3.  $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$  PL, 2

□

**Proposition prf.5.**  $\mathbf{K} \vdash \Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$

*Proof.*

1.  $\varphi \rightarrow (\psi \rightarrow \varphi)$  PL
2.  $\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$  NEC, K.

□

**Proposition prf.6.**  $\mathbf{K} \vdash \neg\Diamond\varphi \rightarrow \Box(\varphi \rightarrow \psi)$

*Proof.*

1.  $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$  PL
2.  $\Box\neg\varphi \rightarrow \Box(\varphi \rightarrow \psi)$  NEC, K
3.  $\neg\neg\Box\neg\varphi \rightarrow \Box(\varphi \rightarrow \psi)$  PL
4.  $\neg\Diamond\varphi \rightarrow \Box(\varphi \rightarrow \psi)$  re-writing.

□

**Proposition prf.7.**  $\mathbf{K} \vdash (\Diamond\varphi \vee \Diamond\psi) \rightarrow \Diamond(\varphi \vee \psi)$

*Proof.*

1.  $\neg(\neg\varphi \rightarrow \psi) \rightarrow \neg\varphi$  PL
2.  $\neg(\varphi \vee \psi) \rightarrow \neg\varphi$  PL, 1
3.  $\Box\neg(\varphi \vee \psi) \rightarrow \Box\neg\varphi$  NEC, K
4.  $\neg\Box\neg\varphi \rightarrow \neg\Box\neg(\varphi \vee \psi)$  PL
5.  $\Diamond\varphi \rightarrow \Diamond(\varphi \vee \psi)$  re-writing
6.  $\Diamond\psi \rightarrow \Diamond(\varphi \vee \psi)$  similarly
7.  $(\Diamond\varphi \vee \Diamond\psi) \rightarrow \Diamond(\varphi \vee \psi)$  PL, 5, 6.

□

**Proposition prf.8.**  $\mathbf{K} \vdash \Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$

*Proof.*

1.  $\neg\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \vee \psi))$  PL
2.  $\Box\neg\varphi \rightarrow (\Box\neg\psi \rightarrow \Box\neg(\varphi \vee \psi))$  NEC, K
3.  $\Box\neg\varphi \rightarrow (\neg\Box\neg(\varphi \vee \psi) \rightarrow \neg\Box\neg\psi)$  PL, 2
4.  $\neg\Box\neg(\varphi \vee \psi) \rightarrow (\Box\neg\varphi \rightarrow \neg\Box\neg\psi)$  PL
5.  $\neg\Box\neg(\varphi \vee \psi) \rightarrow (\neg\neg\Box\neg\psi \rightarrow \neg\Box\neg\varphi)$  PL
6.  $\Diamond(\varphi \vee \psi) \rightarrow (\neg\Diamond\psi \rightarrow \Diamond\varphi)$  re-writing
7.  $\Diamond(\varphi \vee \psi) \rightarrow (\Diamond\psi \vee \Diamond\varphi)$  PL.

□

**Problem prf.1.** Provide **K**-proofs of the following:

1.  $\Diamond\neg\perp \rightarrow (\Box\varphi \rightarrow \Diamond\varphi)$ ;
2.  $\Box(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Box\psi)$ ;
3.  $(\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$ .

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