prf.1 Normal Modal Logics

mod:prf:nor: sec

Not every set of modal formulas can easily be characterized as those formulas derivable from a set of axioms. We want modal logics to be well-behaved. First of all, everything we can derive in classical propositional logic should still be derivable, of course taking into account that the formulas may now contain also $\Box$ and $\Diamond$. To this end, we require that a modal logic contain all tautological instances and be closed under modus ponens.

Definition prf.1. A *modal logic* is a set Σ of modal formulas which

1. contains all tautologies, and
2. is closed under substitution, i.e., if $\varphi \in \Sigma$, and $\theta_1, \ldots, \theta_n$ are formulas, then
$$\varphi[\theta_1/p_1, \ldots, \theta_n/p_n] \in \Sigma,$$
3. is closed under *modus ponens*, i.e., if φ and $\varphi \to \psi \in \Sigma$, then $\psi \in \Sigma$.

In order to use the relational semantics for modal logics, we also have to require that all formulas valid in all modal models are included. It turns out that this requirement is met as soon as all instances of K and DUAL are derivable, and whenever a formula φ is derivable, so is $\Box\varphi$. A modal logic that satisfies these conditions is called *normal*. (Of course, there are also non-normal modal logics, but the usual relational models are not adequate for them.)

Definition prf.2. A modal logic Σ is *normal* if it contains

$$\Box(p \to q) \to (\Box p \to \Box q), \tag{K}$$
$$\Diamond p \leftrightarrow \neg\Box\neg p \tag{DUAL}$$

and is closed under *necessitation*, i.e., if $\varphi \in \Sigma$, then $\Box\varphi \in \Sigma$.

Observe that while tautological implication is "fine-grained" enough to preserve *truth at a world*, the rule NEC only preserves *truth in a model* (and hence also validity in a frame or in a class of frames).

mod:prf:nor: prop:rk

Proposition prf.3. *Every normal modal logic is closed under rule* RK*,*

$$\frac{\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \cdots)}{\Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n) \cdots).}\ \text{RK}$$

Proof. By induction on n: If $n = 1$, then the rule is just NEC, and every normal modal logic is closed under NEC.

Now suppose the result holds for $n - 1$; we show it holds for n.

Assume

$$\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \cdots) \in \Sigma$$

By the induction hypothesis, we have

$$\Box\varphi_1 \to (\Box\varphi_2 \to \cdots \Box(\varphi_{n-1} \to \varphi_n) \cdots) \in \Sigma$$

Since Σ is a normal modal logic, it contains all instances of K, in particular

$$\Box(\varphi_{n-1} \to \varphi_n) \to (\Box\varphi_{n-1} \to \Box\varphi_n) \in \Sigma$$

Using modus ponens and suitable tautological instances we get

$$\Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n) \cdots) \in \Sigma.$$ □

Proposition prf.4. *Every normal modal logic Σ contains $\neg\Diamond\bot$.* mod:prf:nor:prop:notDiamondBot

Problem prf.1. Prove Proposition prf.4.

Proposition prf.5. *Let $\varphi_1, \ldots, \varphi_n$ be formulas. Then there is a smallest modal logic Σ containing all instances of $\varphi_1, \ldots, \varphi_n$.*

Proof. Given $\varphi_1, \ldots, \varphi_n$, define Σ as the intersection of all normal modal logics containing all instances of $\varphi_1, \ldots, \varphi_n$. The intersection is non-empty as $\mathrm{Frm}(\mathcal{L})$, the set of all formulas, is such a modal logic. □

Definition prf.6. The smallest normal modal logic containing $\varphi_1, \ldots, \varphi_n$ is called a *modal system* and denoted by $\mathbf{K}\varphi_1 \ldots \varphi_n$. The smallest normal modal logic is denoted by $\mathbf{K}$.

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Bibliography