Let’s see some more examples of derivability in \( \mathbf{K} \), now using the simplified method introduced in ???.

**Proposition prf.1.** \( \mathbf{K} \vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi) \)

*Proof.*

1. \( \mathbf{K} \vdash (\varphi \rightarrow \psi) \rightarrow (\neg \psi \rightarrow \neg \varphi) \)  
   PL
2. \( \mathbf{K} \vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box \neg \psi \rightarrow \Box \neg \varphi) \)  
   RK, 1
3. \( \mathbf{K} \vdash (\Box \neg \psi \rightarrow \Box \neg \varphi) \rightarrow (\neg \Box \neg \varphi \rightarrow \neg \Box \neg \psi) \)  
   TAUT
4. \( \mathbf{K} \vdash \Box(\varphi \rightarrow \psi) \rightarrow (\neg \Box \neg \psi \rightarrow \neg \Box \neg \varphi) \)  
   PL, 2, 3
5. \( \mathbf{K} \vdash (\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi) \)  
   \( \Diamond \) for \( \neg \Box \neg \).

**Proposition prf.2.** \( \mathbf{K} \vdash \Box \varphi \rightarrow (\Diamond(\varphi \rightarrow \psi) \rightarrow \Diamond \psi) \)

*Proof.*

1. \( \mathbf{K} \vdash \varphi \rightarrow (\neg \psi \rightarrow \neg(\varphi \rightarrow \psi)) \)  
   TAUT
2. \( \mathbf{K} \vdash \Box \varphi \rightarrow (\Box \neg \psi \rightarrow \Box \neg(\varphi \rightarrow \psi)) \)  
   RK, 1
3. \( \mathbf{K} \vdash \Box \varphi \rightarrow (\neg \Box \neg \varphi \rightarrow \neg(\varphi \rightarrow \psi)) \)  
   PL, 2
4. \( \mathbf{K} \vdash \Box \varphi \rightarrow (\Diamond(\varphi \rightarrow \psi) \rightarrow \Diamond \psi) \)  
   \( \Diamond \) for \( \neg \Box \neg \).

**Proposition prf.3.** \( \mathbf{K} \vdash (\Diamond \varphi \lor \Diamond \psi) \rightarrow \Diamond(\varphi \lor \psi) \)

*Proof.*

1. \( \mathbf{K} \vdash \neg(\varphi \lor \psi) \rightarrow \neg \varphi \)  
   TAUT
2. \( \mathbf{K} \vdash \Box \neg(\varphi \lor \psi) \rightarrow \Box \neg \varphi \)  
   RK, 1
3. \( \mathbf{K} \vdash \Box \neg \varphi \rightarrow \neg \Box \neg(\varphi \lor \psi) \)  
   PL, 2
4. \( \mathbf{K} \vdash \varphi \rightarrow \Diamond(\varphi \lor \psi) \)  
   \( \Diamond \) for \( \neg \Box \neg \)
5. \( \mathbf{K} \vdash \Diamond \psi \rightarrow \Diamond(\varphi \lor \psi) \)  
   similarly
6. \( \mathbf{K} \vdash (\Diamond \varphi \lor \Diamond \psi) \rightarrow \Diamond(\varphi \lor \psi) \)  
   PL, 4, 5.

**Proposition prf.4.** \( \mathbf{K} \vdash \Diamond(\varphi \lor \psi) \rightarrow (\Diamond \varphi \lor \Diamond \psi) \)

*Proof.*

1. \( \mathbf{K} \vdash \neg \varphi \rightarrow (\neg \psi \rightarrow \neg(\varphi \lor \psi)) \)  
   TAUT
2. \( \mathbf{K} \vdash \Box \neg \varphi \rightarrow (\Box \neg \psi \rightarrow \Box \neg(\varphi \lor \psi)) \)  
   RK
3. \( \mathbf{K} \vdash \Box \neg \varphi \rightarrow (\neg \Box \neg(\varphi \lor \psi) \rightarrow \neg \Box \neg \psi) \)  
   PL, 2
4. \( \mathbf{K} \vdash \Box \neg(\varphi \lor \psi) \rightarrow (\Box \neg \varphi \rightarrow \Box \neg \psi) \)  
   PL, 3
5. \( \mathbf{K} \vdash \Box \neg \varphi \rightarrow (\Box \neg \psi \rightarrow \Box \neg \varphi) \)  
   PL, 4
6. \( \mathbf{K} \vdash \Diamond(\varphi \lor \psi) \rightarrow (\neg \Diamond \psi \rightarrow \Diamond \varphi) \)  
   \( \Diamond \) for \( \neg \Box \neg \)
7. \( \mathbf{K} \vdash \Diamond(\varphi \lor \psi) \rightarrow (\Diamond \psi \lor \Diamond \varphi) \)  
   PL, 6.
Problem prf.1. Show that the following derivability claims hold:

1. $K \vdash \Diamond \neg \bot \rightarrow (\square \varphi \rightarrow \Diamond \varphi)$;
2. $K \vdash \square (\varphi \lor \psi) \rightarrow (\Diamond \varphi \lor \square \psi)$;
3. $K \vdash (\Diamond \varphi \rightarrow \square \psi) \rightarrow \square (\varphi \rightarrow \psi)$.

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Bibliography