prf.1 Derivations and Modal Systems

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We first define what a derivation is for normal modal logics. Roughly, a derivation is a sequence of formulas in which every element is either (a substitution instance of) one of a number of *axioms*, or follows from previous elements by one of a few inference rules. For normal modal logics, all instances of tautologies, K, and DUAL count as axioms. This results in the modal system **K**, the smallest normal modal logic. We may wish to add additional axioms to obtain other systems, however. The rules are always modus ponens MP and necessitation NEC.

Definition prf.1. Given a modal system $\mathbf{K}\varphi_1 \dots \varphi_n$ and a formula ψ we say that ψ is *derivable* in $\mathbf{K}\varphi_1 \dots \varphi_n$, written $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$, if and only if there are formulas $\chi_1, \dots, \chi_k$ such that $\chi_k = \psi$ and each χ_i is either a tautological instance, or an instance of one of K, DUAL, $\varphi_1, \dots, \varphi_n$, or it follows from previous formulas by means of the rules MP or NEC.

The following proposition allows us to show that $\psi \in \Sigma$ by exhibiting a Σ-proof of ψ.

Proposition prf.2. $\mathbf{K}\varphi_1 \dots \varphi_n = \{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\}$.

Proof. We use induction on the length of derivations to show that $\{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\} \subseteq \mathbf{K}\varphi_1 \dots \varphi_n$.

If the derivation of ψ has length 1, it contains a single formula. That formula cannot follow from previous formulas by MP or NEC, so must be a tautological instance, an instance of K, DUAL, or an instance of one of $\varphi_1, \dots, \varphi_n$. But $\mathbf{K}\varphi_1 \dots \varphi_n$ contains these as well, so $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$.

If the derivation of ψ has length > 1, then ψ may in addition be obtained by MP or NEC from formulas not occurring as the last line in the derivation. If ψ follows from χ and $\chi \to \psi$ (by MP), then χ and $\chi \to \psi \in \mathbf{K}\varphi_1 \dots \varphi_n$ by induction hypothesis. But every modal logic is closed under modus ponens, so $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$. If $\psi \equiv \Box\chi$ follows from χ by NEC, then $\chi \in \mathbf{K}\varphi_1 \dots \varphi_n$ by induction hypothesis. But every normal modal logic is closed under NEC, so $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$.

The converse inclusion follows by showing that $\Sigma = \{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\}$ is a normal modal logic containing all the instances of $\varphi_1, \dots, \varphi_n$, and the observation that $\mathbf{K}\varphi_1 \dots \varphi_n$ is, by definition, the smallest such logic.

1. Every tautology ψ is a tautological instance, so $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$, so Σ contains all tautologies.

2. If $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \chi$ and $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \chi \to \psi$, then $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$: Combine the derivation of χ with that of $\chi \to \psi$, and add the line ψ. The last line is justified by MP. So Σ is closed under modus ponens.

3. If ψ has a derivation, then every substitution instance of ψ also has a derivation: apply the substitution to every formula in the derivation.

(Exercise: prove by induction on the length of derivations that the result is also a correct derivation). So Σ is closed under uniform substitution. (We have now established that Σ satisfies all conditions of a modal logic.)

4. We have $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \mathrm{K}$, so $K \in \Sigma$.

5. We have $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \textsc{dual}$, so $\textsc{dual} \in \Sigma$.

6. If $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \chi$, the additional line $\Box\chi$ is justified by NEC. Consequently, Σ is closed under NEC. Thus, Σ is normal.

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Bibliography