Derivations and Modal Systems

We first define what a derivation is for normal modal logics. Roughly, a derivation is a sequence of formulas in which every element is either (a substitution instance of) one of a number of axioms, or follows from previous elements by one of a few inference rules. For normal modal logics, all instances of tautologies, K, and dual count as axioms. This results in the modal system \( K \), the smallest normal modal logic. We may wish to add additional axioms to obtain other systems, however. The rules are always modus ponens MP and necessitation NEC.

Definition prf.1. Given a modal system \( K \varphi_1 \ldots \varphi_n \) and a formula \( \psi \) we say that \( \psi \) is derivable in \( K \varphi_1 \ldots \varphi_n \), written \( K \varphi_1 \ldots \varphi_n \vdash \psi \), if and only if there are formulas \( \chi_1, \ldots, \chi_k \) such that \( \chi_k = \psi \) and each \( \chi_i \) is either a tautological instance, or an instance of one of K, dual, \( \varphi_1, \ldots, \varphi_n \), or it follows from previous formulas by means of the rules MP or NEC.

The following proposition allows us to show that \( \psi \in \Sigma \) by exhibiting a \( \Sigma \)-proof of \( \psi \).

Proposition prf.2. \( K \varphi_1 \ldots \varphi_n = \{ \psi : K \varphi_1 \ldots \varphi_n \vdash \psi \} \).

Proof. We use induction on the length of derivations to show that \( \{ \psi : K \varphi_1 \ldots \varphi_n \vdash \psi \} \subseteq K \varphi_1 \ldots \varphi_n \).

If the derivation of \( \psi \) has length 1, it contains a single formula. That formula cannot follow from previous formulas by MP or NEC, so must be a tautological instance, an instance of K, dual, or an instance of one of \( \varphi_1, \ldots, \varphi_n \). But \( K \varphi_1 \ldots \varphi_n \) contains these as well, so \( \psi \in K \varphi_1 \ldots \varphi_n \).

If the derivation of \( \psi \) has length > 1, then \( \psi \) may in addition be obtained by MP or NEC from formulas not occurring as the last line in the derivation. If \( \psi \) follows from \( \chi \) and \( \chi \rightarrow \psi \) (by MP), then \( \chi \) and \( \chi \rightarrow \psi \in K \varphi_1 \ldots \varphi_n \) by induction hypothesis. But every modal logic is closed under modus ponens, so \( \psi \in K \varphi_1 \ldots \varphi_n \). If \( \psi \equiv \Box \chi \) follows from \( \chi \) by NEC, then \( \chi \in K \varphi_1 \ldots \varphi_n \) by induction hypothesis. But every normal modal logic is closed under NEC, so \( \psi \in K \varphi_1 \ldots \varphi_n \).

The converse inclusion follows by showing that \( \Sigma = \{ \psi : K \varphi_1 \ldots \varphi_n \vdash \psi \} \) is a normal modal logic containing all the instances of \( \varphi_1, \ldots, \varphi_n \), and the observation that \( K \varphi_1 \ldots \varphi_n \) is, by definition, the smallest such logic.

1. Every tautology \( \psi \) is a tautological instance, so \( K \varphi_1 \ldots \varphi_n \vdash \psi \), so \( \Sigma \) contains all tautologies.

2. If \( K \varphi_1 \ldots \varphi_n \vdash \chi \) and \( K \varphi_1 \ldots \varphi_n \vdash \chi \rightarrow \psi \), then \( K \varphi_1 \ldots \varphi_n \vdash \psi \): Combine the derivation of \( \chi \) with that of \( \chi \rightarrow \psi \), and add the line \( \psi \). The last line is justified by MP. So \( \Sigma \) is closed under modus ponens.

3. If \( \psi \) has a derivation, then every substitution instance of \( \psi \) also has a derivation: apply the substitution to every formula in the derivation.
(Exercise: prove by induction on the length of derivations that the result is also a correct derivation). So $\Sigma$ is closed under uniform substitution. (We have now established that $\Sigma$ satisfies all conditions of a modal logic.)

4. We have $K \varphi_1 \ldots \varphi_n \vdash K$, so $K \in \Sigma$.

5. We have $K \varphi_1 \ldots \varphi_n \vdash \text{dual}$, so $\text{dual} \in \Sigma$.

6. If $K \varphi_1 \ldots \varphi_n \vdash \chi$, the additional line $\Box \chi$ is justified by NEC. Consequently, $\Sigma$ is closed under NEC. Thus, $\Sigma$ is normal. \qed

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Bibliography